Etude de la bornitude des transformées de Riesz sur Lp via le Laplacien de Hodge-de Rham

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Etude de la bornitude des transformées de Riesz sur L_p via le Laplacien de Hodge-de Rham

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Résumé

Cette thèse comporte deux sujets d'étude mêlés. Le premier concerne l'étude de la bornitude sur L^p de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$, où Δ désigne l'opérateur de Laplace-Beltrami (positif). Le second traite de la régularité de Sobolev $W^{1,p}$ de la solution de l 'équation de la chaleur non perturbée. Nous établissons également quelques résultats concernant les transformées de Riesz d'opérateurs de Schrödinger avec un potentiel comportant éventuellement une partie négative.

Dans le cadre de ces travaux, nous nous plaçons sur une variété riemanienne (M, g) complète et non compacte. Nous supposons que M satisfait la propriété de doublement de volume (de constante de doublement égale à D) ainsi qu'une estimation gaussienne supérieure pour son noyau de la chaleur (celui associé à l'opérateur Δ). Nous travaillons avec le laplacien de Hodge-de Rham, noté $\overrightarrow{\Delta}$, agissant sur les 1-formes différentielles de M. En s'appuyant sur la formule de Bochner, liant $\overrightarrow{\Delta}$ à la courbure de Ricci de M, nous assimilons $\overrightarrow{\Delta}$ à un opérateur de Schrödinger à valeurs vectorielles. C'est un argument de dualité, basé sur une formule de commutation algébrique, qui lie l'étude de $\overrightarrow{\Delta}$ à celle de Δ.

Dans une première partie, nous supposons que la partie négative de la courbure de Ricci, notée R_-, est ϵ-sous-critique et nous démontrons que la transformées de Riesz $d\Delta^{-\frac{1}{2}}$ s'étend en un opérateur borné sur L^p pour tout $p \in (1, p_0)$ où $p_0 > 2$ dépend de D et ϵ. Puis, en rajoutant une hypothèse de type intégral sur R_-, nous montrons que R_- est ϵ-sous-critique si et seulement si l'espace des formes harmoniques L^2 de M est trivial.
Dans une seconde partie, nous donnons une preuve alternative au résultat principal de la première partie en considérant des espaces de Hardy. Ensuite nous supposons R_- "suffisamment petite" dans un sens intégral pour assurer la bornitude de $d\Delta^{-\frac{1}{2}}$ sur L^p pour tout $p \in (1, p_2)$ où $p_2 > 3$ dépend de la condition intégrale. Enfin nous montrons deux résultats, l'un positif, l'autre négatif, concernant la bornitude des transformées de Riesz associées à des opérateurs de Schrödinger.
La dernière partie contient quant à elle des estimées L^p du gradient du semi-groupe associé à l'opérateur Δ ainsi que des estimées L^p du semi-groupe associé à l'opérateur $\overrightarrow{\Delta}$. Afin d'obtenir de telles estimées, nous faisons l'hypothèse que R_- est dans une large classe de Kato dans un premier temps, puis nous ajoutons l'hypothèse que R_- est ϵ-sous-critique pour les affiner.

Mots clés : Variété riemanienne, opérateurs de Schrödinger, laplacien de Hodge-de Rham, transformées de Riesz, régularité de Sobolev, noyaux de la chaleur, estimées hors-diagonales.

Boundedness of the Riesz transforms on L^p via the Hodge-de Rham Laplacian

Abstract

This thesis has two main parts. The first one deals with the study of the boundedness on L^p of the Riesz transform $d\Delta^{-\frac{1}{2}}$, where Δ denotes the nonnegative Laplace-Beltrami operator. The second one deals with the Sobolev regularity $W^{1,p}$ of the solution of the heat equation. We also establish some results on the Riesz transforms of Schrödinger operators with a potential possibly having a negative part.

In this work, we consider a complete non-compact Riemannian manifold (M, g). We assume that M satisfies the volume doubling property (with doubling constant equal to D) as well as a Gaussian upper estimate for its heat kernel associated to the operator Δ. We work with the Hodge-de Rham Laplacian $\overrightarrow{\Delta}$, acting on 1-differential forms of M. With the Bochner formula, linking $\overrightarrow{\Delta}$ to the Ricci curvature of M, we see $\overrightarrow{\Delta}$ has a vector-valued Schrödinger operator. It is a duality argument, based on a commutation formula, which links the study of $\overrightarrow{\Delta}$ to the one of Δ.

In a first part, we assume that the negative part of the Ricci curvature R_- is ϵ-subcritical and we prove that the Riesz transform $d\Delta^{-\frac{1}{2}}$ extends to a bounded operator on L^p for all $p \in (1, p_0)$ where $p_0 > 2$ depends on D and ϵ. Then, with an additionnal integral-type assumption on R_-, we prove that R_- is ϵ-subcritical if and only if the space of L^2 harmonic forms of M is trivial.
In a second part, we give an alternative proof to the main result of the first part considering Hardy spaces. Then we suppose that R_- is "small enough" in an integral sense to ensure the boundedness of $d\Delta^{-\frac{1}{2}}$ on L^p for all $p \in (1, p_2)$ where $p_2 > 3$ depends on the integral condition. Finally, we prove two results, one positive, one negative, about the boundedness of the Riesz transforms associated to Schrödinger operators.
In the last part we prove some L^p estimates of the gradient of the semigroup associated to the operator Δ as well as L^p estimates of the semigroup associated to the operator $\overrightarrow{\Delta}$. To prove such estimates, we make the assumption that R_- is in a large Kato class first, then we make the additionnal assumption that R_- is ϵ-subcritical to improve them.

Key words : Riemannian manifolds, Schrödinger operators, Hodge-de Rham Laplacian, Riesz transforms, Sobolev regularity, heat kernels, off-diagonal estimates.

Contents

Chapter 1

Introduction

1.1 Contexte général

Tout au long de cette thèse, (M, g) désigne une variété riemanienne complète non compacte de dimension N. Nous notons respectivement ρ et μ la distance et la mesure riemanienne naturellement associées à la métrique g.

Nous supposons que la variété M satisfait la propriété de doublement de volume, ce qui signifie qu'il existe une constante $C > 0$ telle que

$$v(x, 2r) \leq Cv(x, r) \text{ pour tous } x \in M,\ r \geq 0,$$

où $v(x, r) = \mu(B(x, r))$ est le volume de la boule $B(x, r)$ de centre x et de rayon r. Cette propriété est équivalente à la suivante : il existe deux constantes $C > 0$ et $D > 0$ telles que

$$v(x, \lambda r) \leq C\lambda^D v(x, r) \quad \text{pour tous } x \in M,\ r \geq 0,\ \lambda \geq 1. \tag{D}$$

Nous utiliserons exclusivement cette deuxième formulation. Il est à noter que la constante D n'est pas unique puisque la propriété (D) est vérifiée pour tout $D' > D$. Cependant dans beaucoup de situations il sera préférable de choisir la constante D aussi petite que possible.

Nous considérons Δ l'opérateur de Laplace-Beltrami positif sur M. Il est défini par la méthode des formes sesquilinéaires. La théorie des formes sesquilinéaires nous assure que l'opérateur $-\Delta$ est le générateur d'un semi-groupe $(e^{-t\Delta})_{t\geq 0}$ holomorphe d'angle $\frac{\pi}{2}$ sur $L^2(M)$. Pour plus de détails concernant la définition et les propriétés relatives à l'opérateur Δ, nous renvoyons le lecteur à la Section 2.1.
Nous notons $p_t(x, y)$ le noyau de la chaleur associé à Δ, c'est-à-dire le noyau intégral de $e^{-t\Delta}$ pour tout $t \geq 0$. Nous supposons que $p_t(x, y)$ admet une estimation gaussienne

supérieure, ce qui signifie qu'il existe deux constantes $c, C > 0$ telles que

$$p_t(x,y) \leq \frac{C}{v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right) \text{ pour tous } t > 0,\, x, y \in M. \tag{G}$$

Soit $d\Delta^{-\frac{1}{2}}$ la transformée de Riesz associée à l'opérateur Δ, définie par la formule

$$d\Delta^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d\, e^{-t\Delta} \frac{dt}{\sqrt{t}}$$

où Γ est la fonction Gamma d'Euler et d est la différentiation extérieure sur M.

Dans cette thèse nous étudions la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ sur L^p. Une intégration par parties donne pour tout $u \in \mathcal{C}_0^\infty(M)$

$$\|du\|_2^2 = \int_M |du|^2 d\mu = \int_M \Delta u.u\, d\mu = \|\Delta^{\frac{1}{2}} u\|_2^2.$$

Ainsi la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ s'étend naturellement en un opérateur borné de $L^2(M)$ dans $L^2(\Lambda^1 T^*M)$ où $\Lambda^1 T^*M$ est l'espace des 1-formes différentielles sur M. Nous nous intéressons à la problématique suivante : sous quelles conditions la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ peut-elle s'étendre en un opérateur borné de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour $p \neq 2$? Cette question a suscité de l'intérêt de la part de nombreux chercheurs durant ces dernières décennies.

Commençons par citer les travaux de Coulhon et Duong [20]. Les auteurs ont démontré que sous les hypothèses (D) et (G), la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, 2]$. Leur méthode repose sur la théorie de Calderòn-Zygmund et sur un théorème dû à Duong et McIntosh qui donne des conditions suffisantes pour qu'un opérateur linéaire soit de type faible $(1, 1)$. Ils ont également construit un exemple de variété riemanienne complète non compacte satisfaisant les hypothèses (D) et (G) sur laquelle $d\Delta^{-\frac{1}{2}}$ n'est pas bornée sur L^p pour $p > 2$. Ce contre-exemple montre qu'il est nécessaire de trouver des conditions supplémentaires pour traiter le cas $p > 2$.
Intéressons nous à l'article de Bakry [8]. L'auteur a démontré que sur toute variété riemanienne à courbure de Ricci positive, la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, \infty)$. Cet article nous incite à trouver des conditions géométriques, portant sur la courbure de Ricci, qui assurent la bornitude de $d\Delta^{-\frac{1}{2}}$ sur L^p pour $p > 2$.

C'est dans cet esprit que nous considérons le laplacien de Hodge-de Rham $\vec{\Delta} = d^*d + dd^*$ agissant sur les 1-formes différentielles. Il est défini par la méthode des formes sesquilinéaires (voir Section 2.1).

D'une part, $\overrightarrow{\Delta}$ est lié à l'opérateur de Laplace-Beltrami via la formule de commutation $\overrightarrow{\Delta} d = d\Delta$. Cette formule nous permet, par dualité, de ramener l'étude de la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ sur L^p à celle de la bornitude de la transformée de Riesz $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ sur L^q où $\frac{1}{p}+\frac{1}{q}=1$.
D'autre part, la formule de Bochner nous assure que $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_- = \overline{\Delta} + R_+ - R_-$, où R_+ (resp. R_-) est la partie positive (resp. la partie négative) de la courbure de Ricci de M et ∇ est la connexion de Levi-Civita sur M. Ainsi $\overrightarrow{\Delta}$ peut être assimilé à un opérateur de Schrödinger à valeurs vectorielles, son potentiel étant la courbure de Ricci de M. Notons que définir un opérateur de Schrödinger $\Delta + V_+ - V_-$ demande quelques précautions et des hypothèses sur la partie négative V_- du potentiel. Ce problème est écarté pour $\overrightarrow{\Delta}$ puisque qu'une de ses propriétés fondamentales est d'être un opérateur toujours accrétif. En revanche, la notion de positivité n'ayant plus de sens sur les formes différentielles, certaines techniques concernant l'étude des opérateurs de Schrödinger ne sont pas applicables à $\overrightarrow{\Delta}$. Nous montrons donc dans la suite comment adapter des méthodes relatives à l'étude des transformées de Riesz d'opérateurs de Schrödinger au cas de $\overrightarrow{\Delta}$.
Un des objectifs de cette thèse est de donner des conditions simples sur R_- pour assurer la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ sur L^p pour certains $p > 2$. Nous montrons également quelques résultats concernant les transformées de Riesz d'opérateurs de Schrödinger (dont le potentiel comporte éventuellement une partie négative).

1.2 Etude de $d\Delta^{-\frac{1}{2}}$ et des transformées de Riesz d'opérateurs de Schrödinger

1.2.1 Résultats relatifs dans la littérature

Nous donnons dans cette section un éventail (non exhaustif) de résultats relatifs à l'étude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ ainsi qu'à celle des transformées de Riesz d'opérateurs de Schrödinger.

Nous avons déjà cité ci-dessus l'excellent article de Coulhon et Duong [20] dans lequel les auteurs ont démontré que sous les hypothèses (D) et (G), la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, 2]$.
Quelques années plus tard, les mêmes auteurs dans [21] ont prouvé que si la variété M satisfait les hypothèses (D), (G) et si le noyau de la chaleur $\overrightarrow{p_t}(x, y)$ associé au laplacien de Hodge-de Rham $\overrightarrow{\Delta}$ vérifie une estimation gaussienne supérieure, alors la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, \infty)$. Leur preuve repose sur des arguments de dualité faisant intervenir des inégalités de Littlewood-Paley-Stein et sur l'estimation suivante du gradient du noyau de la chaleur $p_t(x, y)$ associé à

l'opérateur Δ

$$|\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}\, v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right), \forall x,y \in M, \forall t>0.$$

Sikora [48] a proposé une méthode alternative à celle de Coulhon et Duong en démontrant que si la variété M satisfait l'hypothèse (D) et l'estimée L^2

$$\|\overrightarrow{p_t}(x,.)\|_{L^2}^2 \leq \frac{c}{v(x,\sqrt{t})}, \forall t>0, \forall x \in M,$$

alors la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in [2,\infty)$. Sa preuve est basée sur la méthode de l'équation des ondes.

Sous l'hypothèse que M satisfait des estimations de Li-Yau

$$\frac{C'}{v(x,\sqrt{t})} \exp\left(-c'\frac{\rho^2(x,y)}{t}\right) \leq p_t(x,y) \leq \frac{C}{v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right),$$

ou de manière équivalente sous l'hypothèse que M satisfait (D) et une inégalité de Poincaré L^2, Auscher et Coulhon [4] ont prouvé qu'il existe $\epsilon > 0$ tel que $d\Delta^{-\frac{1}{2}}$ est borné de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $2 \leq p < 2+\epsilon$.

Auscher, Coulhon, Duong et Hofmann [5] ont quant à eux caractérisé la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour $p > 2$ à l'aide d'estimées $L^p - L^p$ du gradient du semi-groupe $(e^{-t\Delta})_{t\geq 0}$ sur les variétés riemaniennes satisfaisant des estimations de Li-Yau. Plus précisément les auteurs ont démontré que si le noyau $p_t(x,y)$ vérifie une estimation gaussienne supérieure et inférieure, alors $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour $p \in [2, p_0)$ si et seulement si $\|d\, e^{-t\Delta}\|_{p-p} \leq \frac{C}{\sqrt{t}}$ pour p dans le même intervalle.

Nous passons maintenant en revue quelques résultats faisant intervenir explicitement des hypothèses sur la partie négative R_- de la courbure de Ricci de M.
Inspiré par [21], Devyver [28] a étudié la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ dans le cadre des variétés riemaniennes satisfaisant une inégalité de Sobolev globale de dimension N avec la condition supplémentaire que les boules de grand rayon ont un volume de type polynomial. Il a supposé que la partie négative R_- de la courbure de Ricci de M vérifie la condition $R_- \in L^{\frac{N}{2}-\eta} \cap L^\infty$ pour un certain $\eta > 0$ et que l'espace des 1-formes harmoniques sur M est trivial. Sous ces hypothèses, il a démontré que le noyau $\overrightarrow{p_t}(x,y)$ vérifie une estimation gaussienne supérieure, ce qui implique en particulier la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1,\infty)$. Sans l'hypothèse sur les formes harmoniques, il a aussi démontré que $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$

dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, N)$.

Récemment Carron [16] suppose que les hypothèses (D) et (G) sont vérifiées, que M satisfait la propriété de doublement de volume inverse $v(o, \lambda r) \geq C\lambda^{D'} v(o, r), \forall r > 0, \lambda \geq 1$ pour une certaine constante $D' > 2$ et un point o fixé de M, et que la courbure de Ricci a une décroissance au plus quadratique

$$Ric \geq -\frac{\kappa^2}{\rho^2(o, x)} g.$$

Il démontre alors que la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1, D')$.

Des travaux de Coulhon, Devyver et Sikora [19] ont été portés à notre connaissance très récemment. Les auteurs supposent que les hypothèses (D) et (G) sont vérifiées, que M satisfait la propriété de doublement de volume inverse $v(x, \lambda r) \geq C\lambda^{D'} v(x, r), \forall x \in M, r > 0, \lambda \geq 1$ avec $D' > 2$, et que la partie négative R_- de la courbure de Ricci de M vérifie la condition

$$\sup_{x \in M} \int_{M \setminus B(o,A)} \left(\int_0^\infty p_t(x, y) dt \right) |R_-(y)| d\mu(y) < 1,$$

où o est un point fixé de M et $A > 0$. Sous ces hypothèses, les auteurs démontrent que R_- est ϵ-sous-critique[1] si et seulement si le noyau $\overrightarrow{p_t}(x, y)$ vérifie une estimation gaussienne supérieure.

Enfin citons l'article de Assaad et Ouhabaz [2] dont la première partie de cette thèse s'inspire fortement. Dans cet article, les auteurs étudient la bornitude sur $L^p(M)$ des transformées de Riesz associées à un opérateur de Schrödinger $A = \Delta + V_+ - V_-$. Les auteurs supposent que M satisfait les hypothèses (D) et (G) et qu'il existe $\epsilon \in [0, 1)$ tel que $V_- \leq \epsilon(\Delta + V_+)$ au sens des formes quadratiques. A l'aide d'inégalités de type Gagliardo-Nirenberg, ils commencent par établir des estimations $L^p - L^q$ hors-diagonales pour le semi-groupe e^{-tA} et pour $\sqrt{t}\nabla e^{-tA}$. En utilisant un critère de bornitude de L^p dans $L^{p,\infty}$ pour les opérateurs linéaires, ils déduisent ensuite que la transformée de Riesz $\nabla A^{-\frac{1}{2}}$ est bornée sur L^p pour tout $p \in (1, p_0)$ où $p_0 \in (\frac{2D}{D-2}, \infty]$ dépend de D et ϵ.

1.2.2 Contributions à cette étude

La majeure partie de cette thèse est consacrée à l'étude de la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ ainsi qu'à l'étude des transformées de Riesz d'opérateurs de Schrödinger.

[1] Voir **Définition 1.2.1** ci-après.

Dans le **Chapitre 3**, nous étudions les transformées de Riesz $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ et $d\overrightarrow{\Delta}^{-\frac{1}{2}}$ sous une hypothèse de forte positivité pour $\overrightarrow{\Delta}$. Ce chapitre correspond à l'article [43] à paraitre dans *Mathematische Nachrichten*.
Rappelons que, d'après la formule de Bochner, $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_- := H - R_-$.

Définition 1.2.1. On dit que la partie négative R_- de la courbure de Ricci de M est ϵ-sous-critique s'il existe $\epsilon \in [0,1)$ tel que

$$0 \le (R_-\omega,\omega) \le \epsilon\,(H\omega,\omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M).$$

En adaptant des idées de [2] au cadre des formes différentielles, nous démontrons que si M satisfait les hypothèses (D) et (G) et si R_- est ϵ-sous-critique, alors la transformée de Riesz $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ est bornée de $L^p(\Lambda^1T^*M)$ dans $L^p(M)$ et la transformée de Riesz $d\overrightarrow{\Delta}^{-\frac{1}{2}}$ est bornée de $L^p(\Lambda^1T^*M)$ dans $L^p(\Lambda^2T^*M)$ pour tout $p \in (p_0', 2]$ où $p_0' = \left(\frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}\right)'$ si $D > 2$ et $p_0' = 1$ si $D \le 2$. Utilisant un argument de dualité, on déduit en particulier que la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1T^*M)$ pour tout $p \in (1, p_0)$ où $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ si $D > 2$ et $p_0 = +\infty$ si $D \le 2$.
En ce qui concerne la preuve de ce résultat, nous commençons par montrer que l'opérateur $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_-$ est le générateur d'un semi-groupe analytique uniformément borné sur $L^p(\Lambda^1T^*M)$ pour tout $p \in (p_0', p_0)$. En exploitant des inégalités de type Gagliardo-Nirenberg, nous établissons alors des estimations $L^p - L^q$ hors-diagonales de la forme

$$\|\chi_{C_j(x,r)} e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-q} \le \frac{Ce^{-c\frac{4^j r^2}{t}}}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r})\right)^\beta,$$

pour $e^{-t\overrightarrow{\Delta}}$ ainsi que pour les opérateurs $\sqrt{t}d^*e^{-t\overrightarrow{\Delta}}$ et $\sqrt{t}de^{-t\overrightarrow{\Delta}}$. Enfin nous utilisons un critère de bornitude de L^p dans $L^{p,\infty}$ pour les opérateurs linéaires pour en déduire la bornitude des transformées de Riesz $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ et $d\overrightarrow{\Delta}^{-\frac{1}{2}}$ sur L^p.
Dans une seconde partie, nous supposons en outre qu'il existe $r_1, r_2 > 2$ tels que

$$\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < +\infty.$$

Nous démontrons alors que R_- est ϵ-sous-critique si et seulement si l'espace des formes harmoniques L^2 sur M est trivial.

Le **Chapitre 4** présente des résultats concernant la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ et les transformées de Riesz $dA^{-\frac{1}{2}}$ où $A = \Delta + V_+ - V_-$ est un opérateur de Schrödinger. Ce chapitre correspond à l'article [18] écrit en collaboration avec Peng Chen et El Maati Ouhabaz.

Dans une première partie, nous donnons une preuve alternative au résultat du **Chapitre 3** concernant les transformées de Riesz. Plus précisément nous montrons que si les hypothèses (D) et (G) sont satisfaites et si R_- est α-sous-critique pour un certain $\alpha \in [0,1)$, alors la transformée de Riesz $d^*\vec{\Delta}^{-\frac{1}{2}}$ est bornée de l'espace de Hardy associé $H^p_{\vec{\Delta}}(\Lambda^1 T^*M)$ dans $L^p(M)$ pour tout $p \in [1,2]$. On en déduit que la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (1,p_0)$ où $p_0 > 2$ est défini comme précédemment. La preuve repose sur les estimations $L^p - L^2$ hors-diagonales obtenues au **Chapitre 3** qui permettent d'identifier les espaces $H^p_{\vec{\Delta}}(\Lambda^1 T^*M)$ et $L^p(\Lambda^1 T^*M)$ pour $p \in (p_0', 2]$.
Dans une seconde partie, nous remplaçons l'hypothèse R_- α-sous-critique par la condition intégrale suivante

$$\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_1}}} \right\|_{p_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} < \infty,$$

où $p_1 > 2$ et $p_2 > 3$. Nous démontrons dans ce cas que la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $1 < p < p_2$. La stratégie de la preuve est basée sur la décomposition

$$d\Delta^{-\frac{1}{2}} = d\Delta^{-\frac{1}{2}} - d(\Delta + W)^{-\frac{1}{2}} + d(\Delta + W)^{-\frac{1}{2}} - (\vec{\Delta} + W)^{-\frac{1}{2}}d + (\vec{\Delta} + W)^{-\frac{1}{2}}d,$$

où W est un potentiel lisse à support compact bien choisi. Nous utilisons ensuite des résultats du **Chapitre 3** et des idées de [28] et [2] pour étudier la bornitude sur L^p de chacun des termes.
La dernière partie de ce chapitre est consacrée à l'étude des transformées de Riesz associées à des opérateurs de Schrödinger $A = \Delta + V_+ - V_-$ où $V_+ \in L^1_{loc}(M)$. Nous supposons dans un premier temps que M vérifie les hypothèses (D) et (G) et que R_- satisfait la condition intégrale ci-dessus. Nous supposons également que la partie négative V_- du potentiel est α-sous-critique et vérifie la même condition intégrale que R_-. Sous ces hypothèses, la transformée de Riesz $dA^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $p \in (p_0', \frac{p_0 r}{p_0 + r})$ où $r = inf(p_1, p_2)$. Enfin nous démontrons que si M satisfait (D) ainsi qu'une inégalité de Poincaré L^2, et s'il existe une fonction ϕ bornée sur M telle que $e^{-tA}\phi = \phi$, alors la bornitude de la transformée de Riesz $dA^{-\frac{1}{2}}$ sur L^p pour un $p > \max(2, D)$ entraine la nullité du potentiel $V = V_+ - V_-$. Ce résultat est démontré dans le cas particulier où $V_- = 0$.

1.3 Estimations de $\|\nabla e^{-t\Delta}\|_p$ et $\|e^{-t\vec{\Delta}}\|_p$

La seconde partie de cette thèse est consacrée à l'étude de la régularité de Sobolev $W^{1,p}$ de la solution de l'équation de la chaleur non perturbée sur M. Considérons le problème d'évolution suivant

$$\frac{d}{dt}u + \Delta u = 0,\ \forall t > 0 \text{ et } u(0) = f \in L^p(M).$$

La solution de l'équation de la chaleur ci-dessus est $u(t) = e^{-t\Delta}f$. De plus le semi-groupe $(e^{-t\Delta})_{t\geq 0}$ est un semi-groupe analytique de contractions sur $L^p(M)$ pour tout $p \in (1,\infty)$, c'est-à-dire qu'on a

$$\|e^{-t\Delta}\|_p \leq 1,\ \forall t \geq 0, \forall p \in (1,\infty).$$

On s'intéresse dans cette thèse à la régularité $W^{1,p}$ de la solution, c'est-à-dire à la norme L^p du gradient du semi-groupe $e^{-t\Delta}$. Une manière d'obtenir une estimations de $\|\nabla e^{-t\Delta}\|_p$ est de montrer que la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^*M)$. En effet, quand c'est le cas pour un certain $p \in (1,\infty)$, l'analyticité du semi-groupe $(e^{-t\Delta})_{t\geq 0}$ sur $L^p(M)$ et la formule de Cauchy permettent de montrer que

$$\|\nabla e^{-t\Delta}\|_p \leq \frac{C}{\sqrt{t}},\ \forall t \geq 0.$$

Un des objectifs de cette thèse est de donner des conditions simples sur la partie négative R_- de la courbure de Ricci qui permettent d'obtenir des estimations en temps long de $\|\nabla e^{-t\Delta}\|_p$ lorsque $p > 2$, sans nécessairement utiliser la bornitude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ sur L^p.

Définition 1.3.1. $\tilde{K}^N$ désigne l'ensemble des fonctions f vérifiant la propriété suivante : il existe $\xi > 0$ tel que

$$\sup_{x\in M} \int_M \left(\int_0^\xi p_s(x,y)ds \right) |f(y)| d\mu(y) < 1.$$

Dans le **Chapitre 5**, nous étudions $\|\nabla e^{-t\Delta}\|_p$ et $\|e^{-t\vec{\Delta}}\|_p$ sous les hypothèses (D), (G) et $|R_-| \in \tilde{K}^N$. Nous montrons que pour tout $t \geq 1$ et $p > 2$

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C_p t^{\left(\frac{1}{2}-\frac{1}{p}\right)D - \frac{1}{2}}$$

et que pour tout $t > e$ et $p \in [1,\infty]$

$$\|e^{-t\vec{\Delta}}\|_{p,p} \leq C_p (t\log(t))^{\left|\frac{1}{2}-\frac{1}{p}\right|\frac{D}{2}}.$$

Pour prouver ces résultats, nous montrons au préalable que le noyau $\vec{p_t}(x,y)$ satisfait l'estimation gaussienne suivante

$$|\vec{p_t}(x,y)| \leq C \min\left(1, \frac{t^{\frac{D}{2}}}{v(x,\sqrt{t})}\right) \exp\left(-c\frac{\rho^2(x,y)}{t}\right),$$

où $t \geq 1$ et $x, y \in M$. Sous les mêmes hypothèses, nous démontrons également que pour tout $a > 0$, la transformée de Riesz locale $d(\Delta+a)^{-\frac{1}{2}}$ est bornée de $L^p(M)$ dans $L^p(\Lambda^1 T^* M)$ pour tout $p \in (1, \infty)$.
En outre, en supposant que R_- est ϵ-sous-critique nous améliorons les estimations précédemment obtenues.

Chapter 2

Préliminaires

Dans ce chapitre, nous présentons quelques définitions et résultats constituant une base de connaissances minimale pour bien entamer la lecture des chapitres qui suivent. Les résultats sont énoncés sans leur preuve.

Les trois premières sections visent à définir rigoureusement le laplacien Beltrami et le laplacien de Hodge-de Rham en s'appuyant sur la théorie des formes sesquilinéaires. En outre, nous définirons leurs transformées de Riesz respectives. La quatrième section est quant à elle dédiée aux notions de géométrie riemannienne qui seront utilisées tout au long de cette thèse.

2.1 Théorie des formes sesquilinéaires

Les résultats de cette section sont tirés de [41] et [44].

H désigne un espace de Hilbert sur $\mathbb{K} = \mathbb{R}$ ou $\mathbb{C}$ muni du produit scalaire $(u|v)$ et de sa norme associée $\|u\| = (u|u)^{\frac{1}{2}}$.

2.1.1 Généralités

Définition 2.1.1. Une forme sesquilinéaire $\mathfrak{a}$ définie sur un sous-espace de H, appelé domaine de $\mathfrak{a}$ et noté $\mathcal{D}(\mathfrak{a})$, est une application

$$\mathfrak{a} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \longrightarrow \mathbb{K}$$

$$(u, v) \mapsto \mathfrak{a}(u, v)$$

telle que

1. pour tout $v \in \mathcal{D}(\mathfrak{a})$, l'application $u \mapsto \mathfrak{a}(u, v)$ est linéaire.

2. pour tout $u \in \mathcal{D}(\mathfrak{a})$, l'application $v \mapsto \mathfrak{a}(u,v)$ est antilinéaire.

Définition 2.1.2. Soit $\mathfrak{a} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \to \mathbb{K}$ une forme sesquilinéaire. On dit que

1. $\mathfrak{a}$ est **accrétive** si pour tout $u \in \mathcal{D}(\mathfrak{a})$, $Re\, \mathfrak{a}(u,u) \geq 0$.
2. $\mathfrak{a}$ est **à domaine dense** si $\mathcal{D}(\mathfrak{a})$ est dense dans H.
3. si $\mathfrak{a}$ est accrétive, on dit que $\mathfrak{a}$ est **continue** s'il existe une constante $M \geq 0$ telle que
$$|\mathfrak{a}(u,v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}, \forall u,v \in \mathcal{D}(\mathfrak{a})$$
où $\|u\|_{\mathfrak{a}} = (Re\, \mathfrak{a}(u,u) + \|u\|^2)^{\frac{1}{2}}$.
4. $\mathfrak{a}$ est **fermée** si $(\mathcal{D}(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ est complet.
5. $\mathfrak{a}$ est sectorielle s'il existe une constante $C \geq 0$ telle que
$$|Im\, \mathfrak{a}(u,u)| \leq C Re\, \mathfrak{a}(u,u), \forall u \in \mathcal{D}(\mathfrak{a}).$$

Définition 2.1.3. Soit $\mathfrak{a} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \to \mathbb{K}$ une forme sesquilinéaire. On définit la forme adjointe de $\mathfrak{a}$, notée $\mathfrak{a}^*$ par
$$\mathcal{D}(\mathfrak{a}^*) = \mathcal{D}(\mathfrak{a}) \text{ et } \mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)}.$$
On dit que $\mathfrak{a}$ est **symétrique** si $\mathfrak{a}^* = \mathfrak{a}$, c'est-à-dire si $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ pour tous $u,v \in \mathcal{D}(\mathfrak{a})$.

Proposition 2.1.1. *Soit $\mathfrak{a}$ une forme sesquilinéaire symétrique et fermée. Alors $\mathfrak{a}$ est continue.*

Proposition 2.1.2. *Soit $\mathfrak{a}$ une forme sesquilinéaire sectorielle. Alors $\mathfrak{a}$ est continue et on a*
$$|\mathfrak{a}(u,v)| \leq (1+C)(Re\, \mathfrak{a}(u,u))^{\frac{1}{2}}(Re\, \mathfrak{a}(v,v))^{\frac{1}{2}}, \forall u,v \in \mathcal{D}(\mathfrak{a}).$$

2.1.2 Opérateur associé à une forme sesquilinéaire

Définition 2.1.4. Soit $\mathfrak{a}$ une forme sesquilinéaire à domaine dense, accrétive, continue et fermée. On définit un opérateur A en posant
$$\mathcal{D}(A) = \{u \in \mathcal{D}(\mathfrak{a}) : \exists v \in H, \mathfrak{a}(u,\varphi) = (v|\varphi), \forall \varphi \in \mathcal{D}(\mathfrak{a})\}$$
$$Au = v.$$
A est appelé l'opérateur associé à la forme $\mathfrak{a}$ et $\mathcal{D}(A)$ est appelé domaine de A.

Proposition 2.1.3. *Soit A l'opérateur associé à une forme $\mathfrak{a}$ accrétive, à domaine dense, continue et fermée. Alors A est à domaine dense et pour tout $\lambda > 0$, $\lambda I + A$ est inversible de $\mathcal{D}(A)$ dans H et son inverse $(\lambda I + A)^{-1}$ vérifie*

$$\|\lambda(\lambda I + A)^{-1} f\| \leq \|f\|, \forall f \in H.$$

Proposition 2.1.4. *Soit A l'opérateur associé à une forme $\mathfrak{a}$ accrétive, à domaine dense, continue et fermée. Alors A^* est l'opérateur associé à la forme $\mathfrak{a}^*$.*
En particulier si $\mathfrak{a}$ est une forme symétrique, alors A est auto-adjoint.

Proposition 2.1.5. *Soit $\mathfrak{a}$ une forme sesquilinéaire accrétive et symétrique. Soit A son opérateur associé. Alors il existe un unique opérateur B accrétif et auto-adjoint tel que $B^2 = A$, c'est-à-dire tel que $\mathcal{D}(B^2) = \mathcal{D}(A)$ et $B^2 u = Au$ pour tout $u \in \mathcal{D}(A)$.*
L'opérateur B, noté $A^{\frac{1}{2}}$, vérifie

$$\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(\mathfrak{a}) \text{ et } \mathfrak{a}(u, v) = (A^{\frac{1}{2}} u | A^{\frac{1}{2}} v), \forall u, v \in \mathcal{D}(\mathfrak{a}).$$

2.1.3 Semi-groupe associé à une forme sesquilinéaire

Comme conséquence de la **Proposition 2.1.3** et du théorème de Hille-Yosida, nous avons le résultat suivant.

Théorème 2.1.6. *Soit $\mathfrak{a}$ une forme sesquilinéaire à domaine dense, accrétive, continue et fermée. Soit A son opérateur associé. Alors $-A$ est le générateur d'un semi-groupe $(e^{-tA})_{t \geq 0}$ de contractions fortement continu sur H.*

En rajoutant une hypothèse de sectorialité sur $\mathfrak{a}$ on récupère l'analyticité du semi-groupe.

Théorème 2.1.7. *Soit $\mathfrak{a}$ une forme sesquilinéaire sectorielle*

$$|Im\, \mathfrak{a}(u, u)| \leq C Re\, \mathfrak{a}(u, u), \forall u \in \mathcal{D}(\mathfrak{a})$$

et fermée. Soit A son opérateur associé. Alors $-A$ est le générateur d'un semi-groupe holomorphe d'angle $\frac{\pi}{2} - Arctan(C)$. De plus

$$\|e^{-zA}\|_{\mathcal{L}(H)} \leq 1, \forall z \in \Sigma\left(\frac{\pi}{2} - Arctan(C)\right)$$

où $\Sigma\left(\frac{\pi}{2} - Arctan(C)\right) = \{z \in \mathbb{C}, |arg(z)| \leq \frac{\pi}{2} - Arctan(C)$.

En particulier si $\mathfrak{a}$ est symétrique, alors $\mathfrak{a}$ est sectorielle (avec $C = 0$) et ainsi $-A$ est le générateur d'un semi-groupe holomorphe d'angle $\frac{\pi}{2}$ et

$$\|e^{-zA}\|_{\mathcal{L}(H)} \leq 1, \forall z \in \Sigma\left(\frac{\pi}{2}\right).$$

2.2 Opérateur de Laplace-Beltrami

2.2.1 Définition

Soit M une variété riemannienne complète non-compacte. L'opérateur de Laplace-Beltrami (positif), noté Δ, est l'opérateur associé à la forme sesquilinéaire $\mathfrak{a}$ définie sur l'espace de Hilbert $L^2(M)$ par

$$\mathfrak{a}(u,v) = \int_M < du(x), dv(x) >_x \, d\mu(x)$$

$$\mathcal{D}(\mathfrak{a}) = \overline{\mathcal{C}_0^\infty(M)}^{\|.\|_{\mathfrak{a}}}$$

où $\|u\|_{\mathfrak{a}} = (\mathfrak{a}(u,u) + \|u\|_2)^{\frac{1}{2}}$. Formellement $\Delta = d^*d = -div \circ \nabla$. L'opérateur $-\Delta$ est le générateur d'un semi-groupe $(e^{-t\Delta})_{t\geq 0}$ holomorphe d'angle $\frac{\pi}{2}$ sur $L^2(M)$.

2.2.2 Transformée de Riesz associée

La transformée de Riesz de l'opérateur de Laplace-Beltrami est l'opérateur $d\Delta^{-\frac{1}{2}}$ défini par

$$d\Delta^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d\, e^{-t\Delta} \frac{dt}{\sqrt{t}}$$

où Γ est la fonction Gamma d'Euler. Une intégration par parties donne pour tout $u \in \mathcal{C}_0^\infty(M)$

$$\|du\|_2^2 = \int_M |du|^2 d\mu = \int_M \Delta u.u \, d\mu = \mathfrak{a}(u,u) = \|\Delta^{\frac{1}{2}} u\|_2^2.$$

Ainsi la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ s'étend naturellement en un opérateur borné de $L^2(M)$ dans $L^2(\Lambda^1 T^*M)$ où $\Lambda^1 T^*M$ est l'espace des 1-formes différentielles sur M.

2.3 Laplacien de Hodge-de Rham

2.3.1 Définition

Pour tout $k \in \mathbb{N}$, notons $\Lambda^k T^*M$ l'espace des k-formes différentielles définies sur M (avec $\Lambda^0 T^*M$ désignant par convention l'espace des fonctions définies sur M). Soit $d_k : \Lambda^k T^*M \to \Lambda^{k+1} T^*M$ l'opérateur de différentiation extérieure sur M. On définit le laplacien de Hodge-de Rham $\overrightarrow{\Delta}$ comme étant l'opérateur associé à la forme sesquilinéaire $\overrightarrow{\mathfrak{a}}$ définie sur l'espace de Hilbert $L^2(\Lambda^1 T^*M)$ par

$$\overrightarrow{\mathfrak{a}}(\omega,\eta) = \int_M < d_1\omega(x), d_1\eta(x) >_x \, d\mu(x) + \int_M d_0^*\omega(x).d_0^*\eta(x) d\mu(x)$$

$$\mathcal{D}(\overrightarrow{\mathfrak{a}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|.\|_{\overrightarrow{\mathfrak{a}}}}$$

où $\|\omega\|_{\overrightarrow{\mathfrak{a}}} = (\overrightarrow{\mathfrak{a}}(\omega,\omega) + \|\omega\|_2^2)^{\frac{1}{2}}$. Formellement $\overrightarrow{\Delta} = d_1^* d_1 + d_0 d_0^*$. L'opérateur $-\overrightarrow{\Delta}$ est le générateur d'un semi-groupe $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ holomorphe d'angle $\frac{\pi}{2}$ sur $L^2(\Lambda^1 T^* M)$.

2.3.2 Transformées de Riesz associées

On associe à l'opérateur $\overrightarrow{\Delta}$ les deux transformées de Riesz $d_0^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ et $d_1 \overrightarrow{\Delta}^{-\frac{1}{2}}$ définies respectivement par les formules

$$d_0^* \overrightarrow{\Delta}^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d_0^* e^{-t\overrightarrow{\Delta}} \frac{dt}{\sqrt{t}}$$

$$d_1 \overrightarrow{\Delta}^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d_1 e^{-t\overrightarrow{\Delta}} \frac{dt}{\sqrt{t}}.$$

Comme pour l'opérateur de Laplace-Beltrami, une intégration par parties montre que les transformées de Riesz $d_0^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ et $d_1 \overrightarrow{\Delta}^{-\frac{1}{2}}$ s'étendent en des opérateurs bornés de $L^2(\Lambda^1 T^* M)$ dans $L^2(M)$ et de $L^2(\Lambda^1 T^* M)$ dans $L^2(\Lambda^2 T^* M)$ respectivement.

Dans ce qui suit, afin de ne pas alourdir les notations, on écrira abusivement d au lieu de d_k.

2.3.3 Formule de commutation

Pour relier l'étude de la transformée de Riesz $d\Delta^{-\frac{1}{2}}$ à celle de la transformée de Riesz $d^* \overrightarrow{\Delta}^{-\frac{1}{2}}$, nous utilisons la formule de commutation suivante

$$\overrightarrow{\Delta} d = (dd^* + d^* d) d = d(d^* d) = d\Delta.$$

Cette dernière implique une formule de commutation pour les semi-groupes associés à $-\Delta$ et $-\overrightarrow{\Delta}$

$$e^{-t\overrightarrow{\Delta}} d = d e^{-t\Delta}, \forall t \geq 0,$$

de sorte que $d^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ est l'adjoint de $d\Delta^{-\frac{1}{2}}$. Ainsi pour montrer que $d\Delta^{-\frac{1}{2}}$ est borné sur L^p, il suffit de montrer que $d^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ est borné sur $L^{p'}$ avec p' tel que $\frac{1}{p} + \frac{1}{p'} = 1$.

2.4 Eléments de géométrie riemannienne

2.4.1 Variétés riemaniennes

Définition 2.4.1. Une variété riemanienne (M, g) est une variété lisse M dotée d'une famille $g = (g_x)_{x \in M}$ de produit scalaire sur l'espace tangent TM avec l'application $M \longrightarrow T_x^* M \otimes T_x^* M$, $x \mapsto g_x$ de classe $\mathcal{C}^\infty$. On appelle la famille g une métrique riemanienne sur M.

Remarque 2.4.1. Soit $v, w \in T_xM$ et soit $(x^1, x^2, \cdots, x^n)$ un système de coordonnées autour de x. Il existe des scalaires $(\alpha_i)_i$ et $(\beta_i)_i$ tels que

$$v = \sum_i \alpha_i \, \partial x^i, \ w = \sum_j \beta_j \, \partial x^j,$$

où $(\partial x^i)_i$ est une base orthonormée directe de T_xM. On a alors

$$g_x(v, w) = g_x(\sum_i \alpha_i \, \partial x^i, \sum_j \beta_j \, \partial x^j) = \sum_{i,j} \alpha_i \beta_j g_x(\partial x^i, \partial x^j).$$

En notant $g_{ij}(x) = g_x(\partial x^i, \partial x^j)$, on trouve comme expression de g

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j,$$

où $dx^i = \partial x^{i*} \in T_x^*M$.

Remarque 2.4.2. Si $(E, < .,. >)$ est un espace euclidien, il est isomorphe à son dual E^* via l'application $E \longrightarrow E^*$, $x \mapsto x^*$ où $x^*(y) =< x, y >$. Ainsi E^* peut être muni d'un produit scalaire, aussi noté $< .,. >$, défini par $< x^*, y^* >=< x, y >$. En particulier, une métrique riemanienne g sur une variété induit un produit scalaire sur chaque espace cotangent T_x^*M, noté encore g_x. On pose ensuite

$$g^{ij} = g(dx^i, dx^j).$$

2.4.2 Distance associée à une métrique riemanienne

Définition 2.4.2. Soit (M, g) une variété riemanienne. On définit la longueur d'une courbe $\gamma : [0, 1] \longrightarrow M$ par

$$l(\gamma) = \int_0^1 g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{\frac{1}{2}} dt.$$

Définition 2.4.3. Soient $x, y \in M$. On définit la distance entre x et y par

$$\rho(x, y) = \inf_{\gamma \, \mathcal{C}^\infty \, par\, morceaux} \{l(\gamma)/\gamma(0) = x, \gamma(1) = y\}.$$

Le couple (M, ρ) est un espace métrique.

2.4.3 Mesure associée à une métrique riemanienne

Définition 2.4.4. Soit (M, g) une variété riemanienne. On définit une mesure μ sur M exprimée en coordonnées locales en posant

$$d\mu = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n,$$

où $\det g = \det(g_{ij})_{i,j}$ et $\wedge$ est le produit extérieur sur M. Ainsi pour tout ensemble mesurable A de M, la mesure de A est donnée par $\mu(A) = \int_A d\mu(x)$.

2.4.4 Connexion de Levi-Civita et formule de Bochner

Dans cette section nous définissons la connexion de Levi-Civita associée à une variété riemanienne (M, g). Celle-ci permet de donner un sens à la dérivée de tenseurs et, plus particulièrement dans le cas qui nous intéresse, à la dérivée de 1-formes différentielles. De plus il est à noter que cet opérateur restreint aux fonctions coincide avec la différentiation extérieure. La définition de la connexion de Levi-Civita nous permet également d'énoncer la formule de Bochner qui relie le laplacien de Hodge-de Rham $\overrightarrow{\Delta}$ avec la courbure de Ricci. Cette formule constitue le point de départ de notre étude.

Les coefficients suivants interviennent dans la définition de la connexion de Levi-Civita.

Définition 2.4.5. On définit les symboles de Christoffel (en coordonnés locales) par

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

Dans la suite $\Gamma(TM)$ désigne l'ensemble des sections de TM dans TM, c'est-à-dire l'ensemble des champs de vecteurs $\mathcal{C}^\infty$ de TM dans lui-même.

Définition 2.4.6. Soit (M, g) une variété riemanienne. La connexion de Levi-Civita (ou connexion canonique) associée à (M, g) est l'opérateur $\nabla : \Gamma(TM) \longrightarrow T^*M \otimes \Gamma(TM)$ défini par les conditions

(i) $\nabla \partial x^k = \sum_j dx^j \otimes \left(\sum_i \Gamma^i_{jk} \partial x^i \right)$.

(ii) $\nabla X(\lambda v + w) = \lambda \nabla X(v) + \nabla X(w)$ pour tout $X \in \Gamma(TM)$ et tous $v, w \in TM$.

(iii) $\nabla(fX)(v) = f \nabla X(v) + df(v)X$ pour tout $X \in \Gamma(TM)$ et toute fonction $f : M \to \mathbb{R}$.

Cette définition s'étend aux 1-formes différentielles.

Définition 2.4.7. Soit (M, g) une variété riemanienne. La connexion de Levi-Civita ∇ s'étend en un opérateur, noté encore ∇, de $\Lambda^1 T^*M$ dans $\Lambda^2 T^*M$ par les propriétés suivantes

(i) $\nabla dx^i = -\sum_{j,k} \Gamma^i_{jk} dx^j \otimes dx^k$.

(ii) $\nabla(\lambda\omega + \eta) = \lambda\nabla\omega + \nabla\eta$ pour tous $\omega, \eta \in \Lambda^1 T^*M$.

(iii) $\nabla f = df$ pour tout $f \in \mathcal{C}^\infty(M)$.

(iv) $\nabla(a\omega) = a\nabla\omega + \nabla a \otimes \omega$ pour toute fonction $a : M \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ et tout $\omega \in \Lambda^1 T^*M$.

Remarque 2.4.3. 1) Les deux définitions précédentes ne dépendent pas du choix des coordonnées locales.
2) Il est possible d'étendre la connexion de Levi-Civita à tous les (p, q)-tenseurs, $p, q \in \mathbb{N}$.

Nous pouvons à présent énoncer la formule de Bochner.

Théorème 2.4.4 (Formule de Bochner)**.** *Si ∇^* désigne l'adjoint formel de ∇ sur $L^2(\Lambda^1 T^*M)$ et Ric la courbure de Ricci de (M, g), nous avons*

$$\overrightarrow{\Delta} = \nabla^*\nabla + Ric.$$

Cette formule nous permet, en particulier, de considérer le laplacien de Hodge-de Rham $\overrightarrow{\Delta}$ comme un opérateur de Schrödinger à valeurs vectorielles. Cette idée est à la base de toute notre étude.

Chapter 3

Riesz transforms of the Hodge-de Rham Laplacian on Riemannian manifolds

Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property. Let $\vec{\Delta}$ be the Hodge-de Rham Laplacian acting on 1-differential forms. According to the Bochner formula, $\vec{\Delta} = \nabla^*\nabla + R_+ - R_-$ where R_+ and R_- are respectively the positive and negative part of the Ricci curvature and ∇ is the Levi-Civita connection. We study the boundedness of the Riesz transform $d^*\vec{\Delta}^{-\frac{1}{2}}$ from $L^p(\Lambda^1T^*M)$ to $L^p(M)$ and of the Riesz transform $d\vec{\Delta}^{-\frac{1}{2}}$ from $L^p(\Lambda^1T^*M)$ to $L^p(\Lambda^2T^*M)$. We prove that, if the heat kernel on functions $p_t(x,y)$ satisfies a Gaussian upper bound and if the negative part R_- of the Ricci curvature is ϵ-sub-critical for some $\epsilon \in [0,1)$, then $d^*\vec{\Delta}^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1T^*M)$ to $L^p(M)$ and $d\vec{\Delta}^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1T^*M)$ to $L^p(\Lambda^2T^*M)$ for $p \in (p_0', 2]$ where $p_0 > 2$ depends on ϵ and on a constant appearing in the volume doubling property. A duality argument gives the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1T^*M)$ for $p \in [2, p_0)$ where Δ is the non-negative Laplace-Beltrami operator. We also give a condition on R_- to be ϵ-sub-critical under both analytic and geometric assumptions.

3.1 Introduction and main results

Let (M,g) be a complete non-compact Riemannian manifold of dimension N, where g denotes a Riemannian metric on M ; that is, g is a family of smoothly varying positive definite inner products g_x on the tangent space T_xM for each $x \in M$. Let ρ and μ be the Riemannian distance and measure associated with g respectively. We suppose that M satisfies the volume doubling property, that is, there exists constants $C, D > 0$ such that

$$v(x,\lambda r) \leq C\lambda^D v(x,r),\ \forall x \in M, \forall r \geq 0, \forall \lambda \geq 1, \tag{D}$$

where $v(x,r) = \mu(B(x,r))$ denotes the volume of the ball $B(x,r)$ of center x and radius r. We also say that M is of homogeneous type. This property is equivalent to the existence of a constant $C > 0$ such that

$$v(x, 2r) \leq C v(x,r),\ \forall x \in M, \forall r \geq 0.$$

Let Δ be the non-negative Laplace-Beltrami operator and let $p_t(x,y)$ be the heat kernel of M, that is, the kernel of the semigroup $(e^{-t\Delta})_{t\geq 0}$ acting on $L^2(M)$. We say that the heat kernel $p_t(x,y)$ satisfies a Gaussian upper bound if there exist constants $c, C > 0$ such that

$$p_t(x,y) \leq \frac{C}{v(x,\sqrt{t})} \exp(-c\frac{\rho^2(x,y)}{t}), \forall t > 0, \forall x, y \in M. \tag{G}$$

Let $d\Delta^{-\frac{1}{2}}$ be the Riesz transform of the operator Δ where d denotes the exterior derivative on M. Since we have by integration by parts

$$\|d\, f\|_2 = \|\Delta^{\frac{1}{2}} f\|_2, \forall f \in \mathcal{C}_0^\infty(M),$$

the Riesz transform $d\Delta^{-\frac{1}{2}}$ extends to a bounded operator from $L^2(M)$ to $L^2(\Lambda^1 T^* M)$, where $\Lambda^1 T^* M$ denotes the space of 1-forms on M. An interesting question is whether $d\Delta^{-\frac{1}{2}}$ can be extended to a bounded operator from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \neq 2$. This problem has attracted attention in recent years. We recall some known results.

It was proved by Coulhon and Duong [20] that under the assumptions (D) and (G), the Riesz transform $d\Delta^{-\frac{1}{2}}$ is of weak-type $(1,1)$ and then bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1,2]$. In addition, they gave an example of a complete non-compact Riemannian manifold satisfying (D) and (G) for which $d\Delta^{-\frac{1}{2}}$ is unbounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p > 2$. This manifold consists into two copies of $\mathbb{R}^2$ glued together around the unit circle. See also the article of Carron, Coulhon and Hassell [17] for further results on manifolds with Euclidean ends or the article of Guillarmou and Hassell [36] for complete non-compact and asymptotically conic Riemannian manifolds.

The counter-example in [20] shows that additional assumptions are needed to treat the case $p > 2$. In 2003, Coulhon and Duong [21] proved that if the manifold M satisfies (D), (G) and the heat kernel $\overrightarrow{p_t}(x,y)$ associated with the Hodge-de Rham Laplacian $\overrightarrow{\Delta}$ acting on 1-forms satisfies a Gaussian upper bound, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1,\infty)$. The proof is based on duality arguments and on the following estimate of the gradient of the heat kernel of M

$$|\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}\, v(x,\sqrt{t})} e^{-c\frac{\rho^2(x,y)}{t}}, \forall x, y \in M, \forall t > 0,$$

which is a consequence of the relative Faber-Krahn inequalities satisfied by M and the Gaussian estimates satisfied by $e^{-t\overrightarrow{\Delta}}$.

In 1987, Bakry [8] proved that if the Ricci curvature is non-negative on M, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1,\infty)$. The proof uses probabilistic techniques and the domination

$$|e^{-t\overrightarrow{\Delta}}\omega| \leq e^{-t\Delta}|\omega|, \forall t>0, \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M).$$

In this particular setting, (G) is satisfied, and hence the heat kernel $\overrightarrow{p_t}(x,y)$ satisfies a Gaussian upper bound too. Thus the result of Bakry can be recovered using the arguments of Coulhon and Duong [21]. Note that the result of Bakry does not contredict the counter-example of Coulhon and Duong since the gluing of two copies of $\mathbb{R}^2$ creates some negative curvature.

In 2004, Sikora [48] gave an alternative proof to the previous result of Coulhon and Duong showing that if the manifold M satisfies (D) and the estimate

$$\|\overrightarrow{p_t}(x,.)\|_{L^2}^2 \leq \frac{c}{v(x,\sqrt{t})}, \forall t>0, \forall x \in M,$$

then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in [2,\infty)$. The proof is based on the method of the wave equation.

Auscher, Coulhon, Duong and Hofmann [5] characterized the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $p>2$ in terms of $L^p - L^p$ estimates of the gradient of the heat semigroup when the Riemannian manifold M satisfies Li-Yau estimates. More precisely, they proved that if $p_t(x,y)$ satisfies both Gaussian upper and lower bounds, then $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $p \in [2,p_0)$ if and only if $\|d\, e^{-t\Delta}\|_{p-p} \leq \frac{C}{\sqrt{t}}$ for p in the same interval.

Inspired by [21], Devyver [28] proved a boundedness result for the Riesz transform $d\Delta^{-\frac{1}{2}}$ in the setting of Riemannian manifolds satisfying a global Sobolev inequality of dimension N with an additional assumption that balls of great radius have a polynomial volume growth. It is known in this setting that both (D) and (G) are satisfied. He assumed that the negative part R_- of the Ricci curvature satisfies the condition $R_- \in L^{\frac{N}{2}-\eta} \cap L^\infty$ for some $\eta>0$ and that there is no harmonic 1-form on M. Under these assumptions, he showed that $\overrightarrow{p_t}(x,y)$ satisfies a Gaussian upper bound which implies the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1,\infty)$. Without the assumption on harmonic 1-forms, it is also proved in [28] that $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1,N)$.

In this article, we study the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $p>2$ assuming M satisfies the volume doubling property (D) and $p_t(x,y)$ satisfies a Gaussian upper bound (G). Before stating our results, we recall the Bochner formula $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_- =: H - R_-$, where R_+ (resp. R_-) is the positive part (resp.

negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M. This formula allows us to consider the Hodge-de Rham Laplacian as a "generalized" Schrödinger operator acting on 1-forms. We then make a standard assumption on the negative part R_- ; namely, we suppose that R_- is ϵ-sub-critical, which means that for a certain $\epsilon \in [0,1)$

$$0 \leq (R_-\omega, \omega) \leq \epsilon\,(H\omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M). \tag{S-C}$$

For further information on condition (S-C), see [26] or [23] and the references therein.
Under these assumptions, we prove the following results.

Theorem 3.1.1. *Assume that (D), (G) and (S-C) are satisfied. Then the Riesz transform $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^*M)$ to $L^p(M)$ and the Riesz transform $d\overrightarrow{\Delta}^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^*M)$ to $L^p(\Lambda^2 T^*M)$ for all $p \in (p_0', 2]$ where, $p_0' = \left(\frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}\right)'$ if $D > 2$ and $p_0' = 1$ if $D \leq 2$.*

Here and throughout this paper, p_0' denotes the conjugate of p_0.
Concerning the Riesz transform on functions, we have the following result.

Corollary 3.1.2. *Assume that (D), (G) and (S-C) are satisfied. Then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, p_0)$ where, $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ if $D > 2$ and $p_0 = +\infty$ if $D \leq 2$.*
In particular, the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, \frac{2D}{D-2})$ if $D > 2$ and all $p \in (1, +\infty)$ if $D \leq 2$.

In these results, the constant D is as in (D) and ϵ is as in (S-C). Of course, we take the smallest possible D and ϵ for which (D) and (S-C) are satisfied. The operator d denotes the exterior derivative acting from the space of 1-forms to the space of 2-forms or from the space of functions to the space of 1-forms according to the context. The operator d^* denotes the L^2-adjoint of the exterior derivative d, the latter acting from the space of functions to the space of 1-forms.

Proof of Corollary 3.1.2. According to the commutation formula $\overrightarrow{\Delta} d = d\Delta$, we see that the adjoint operator of $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is exactly $d\Delta^{-\frac{1}{2}}$. Then the case $p \in [2, p_0)$ in **Corollary 3.1.2** is an immediate consequence of **Theorem 3.1.1**. We recall that the case $p \in (1, 2]$ is proved in [20] under the assumptions (D) and (G). □

Before stating our next result, we set

$$Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) := \{\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}) : \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), (\omega, \overrightarrow{\Delta}\eta) = 0\},$$

where $\mathcal{D}(\overrightarrow{\mathfrak{h}})$ is the domain of the closed sesquilinear form $\overrightarrow{\mathfrak{h}}$ whose associated operator is H (see the next section for the definition of $\overrightarrow{\mathfrak{h}}$). We prove the following.

Theorem 3.1.3. *Assume that both (D) and (G) are satisfied. In addition, suppose that for some* $r_1, r_2 > 2$

$$\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < +\infty \tag{3.1}$$

and

$$Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) = \{0\}. \tag{3.2}$$

Then there exists $\epsilon \in [0,1)$ *such that the Riesz transform* $d\Delta^{-\frac{1}{2}}$ *is bounded from* $L^p(M)$ *to* $L^p(\Lambda^1 T^*M)$ *for all* $p \in (1, p_0)$ *where* $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ *if* $D > 2$ *and* $p_0 = +\infty$ *if* $D \leq 2$. *In particular, the Riesz transform* $d\Delta^{-\frac{1}{2}}$ *is bounded from* $L^p(M)$ *to* $L^p(\Lambda^1 T^*M)$ *for all* $p \in (1, \frac{2D}{D-2})$ *if* $D > 2$ *and all* $p \in (1, +\infty)$ *if* $D \leq 2$.

We emphasize that in **Theorem 3.1.1**, **Corollary 3.1.2** and **Theorem 3.1.3**, neither a global Sobolev-type inequality nor any estimates on $\nabla_x p_t(x,y)$ or $\|\overrightarrow{p_t}(x,y)\|$ are assumed.

Condition (3.1) was introduced by Assaad and Ouhabaz [2]. Note that if $v(x,r) \simeq r^N$, then (3.1) means that $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$. In addition, we show that if the quantity

$$\|R_-^{\frac{1}{2}}\|_{vol} := \int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}}$$

is small enough, then R_- is ϵ-sub-critical for some $\epsilon \in [0,1)$ depending on $\|R_-^{\frac{1}{2}}\|_{vol}$ and on the constants appearing in (D) and (G).

Condition (3.2) was also considered by Devyver [28]. By definition, the space $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta})$ is precisely the space of L^2 harmonic 1-forms. See the last section for more details.

The proof of **Theorem 3.1.1** uses similar techniques as in Assaad and Ouhabaz [2] where the Riesz transforms of Schrödinger operators $\Delta + V$ are studied for signed potentials. However the arguments from [2] need substantial modifications, since our Schrödinger operator $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_-$ is a vector-valued operator. In particular we cannot use any sub-Markovian property, as is used in [2].

In Section 3.2, we discuss some preliminaries which are necessary for the main proofs. In Section 3.3, we prove that under the assumptions (D), (G) and (S-C), the operator $\overrightarrow{\Delta}$ generates a uniformly bounded analytic semigroup on $L^p(\Lambda^1 T^*M)$ for all $p \in (p_0', p_0)$ where p_0 is as in **Theorem 3.1.1**. Section 3.4 is devoted to the proof of **Theorem 3.1.1**. Here we use the results of Section 3.3. In the last section we prove **Theorem 3.1.3** ; one of the main step is to prove that if the manifold M satisfies condition (3.1), then R_- satisfies (S-C) if and only if condition (3.2) is satisfied. Here the constant ϵ appearing in (S-C) is the L^2-L^2 norm of the operator $H^{-\frac{1}{2}}R_-H^{-\frac{1}{2}}$.

3.2 Preliminaries

For all $x \in M$ we denote by $< .,. >_x$ the inner product in the tangent space T_xM, in the cotangent space T_x^*M or in the tensor product $T_x^*M \otimes T_x^*M$. By $(.,.)$ we denote the inner product in the Lebesgue space $L^2(M)$ of functions, in the Lebesgue space $L^2(\Lambda^1 T^*M)$ of 1-forms or in the Lebesgue space $L^2(\Lambda^2 T^*M)$ of 2-forms. By $\|.\|_p$ we denote the usual norm in $L^p(M)$, $L^p(\Lambda^1 T^*M)$ or $L^p(\Lambda^2 T^*M)$ and by $\|.\|_{p-q}$ the norm of operators from L^p to L^q (according to the context). The spaces $\mathcal{C}_0^\infty(M)$ and $\mathcal{C}_0^\infty(\Lambda^1 T^*M)$ denote respectively the space of smooth functions and smooth 1-forms with compact support on M. We denote by d the exterior derivative on M and d^* its L^2-adjoint operator. According to the context, the operator d acts from the space of functions on M to $\Lambda^1 T^*M$ or from $\Lambda^1 T^*M$ to $\Lambda^2 T^*M$. If E is a subset of M, χ_E denotes the indicator function of E.

For $\omega, \eta \in \Lambda^1 T^*M$ and for $x \in M$, we denote by $\omega(x) \otimes \eta(x)$ the tensor product of the linear forms $\omega(x)$ and $\eta(x)$. The inner product on the cotangent space T_x^*M induces an inner product on each tensor product $T_x^*M \otimes T_x^*M$ given by

$$< \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) >_x = < \omega_1(x), \omega_2(x) >_x < \eta_1(x), \eta_2(x) >_x,$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$.

We consider Δ the non-negative Laplace-Beltrami operator acting on $L^2(M)$ and $p_t(x,y)$ the heat kernel of M, that is, the integral kernel of the semigroup $e^{-t\Delta}$.

We consider the Hodge-de Rham Laplacian $\overrightarrow{\Delta} = d^*d + dd^*$ acting on $L^2(\Lambda^1 T^*M)$. The Bochner formula says that $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_-$, where R_+ (resp. R_-) is the positive part (resp. negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M. It allows us to look at $\overrightarrow{\Delta}$ as a "generalized" Schrödinger operator with signed vector potential $R_+ - R_-$.

We define the self-adjoint operator $H = \nabla^*\nabla + R_+$ on $L^2(\Lambda^1 T^*M)$ using the method of sesquilinear forms. That is, for all $\omega, \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, we set

$$\overrightarrow{\mathfrak{h}}(\omega, \eta) = \int_M < \nabla\omega(x), \nabla\eta(x) >_x \, d\mu + \int_M < R_+(x)\omega(x), \eta(x) >_x \, d\mu,$$

$$\text{and } \mathcal{D}(\overrightarrow{\mathfrak{h}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|.\|_{\overrightarrow{\mathfrak{h}}}},$$

where $\|\omega\|_{\overrightarrow{\mathfrak{h}}} = \sqrt{\overrightarrow{\mathfrak{h}}(\omega, \omega) + \|\omega\|_2^2}$.

We say that R_- is ϵ-sub-critical if for a certain constant $0 \leq \epsilon < 1$

$$0 \leq (R_- \omega, \omega) \leq \epsilon\, (H\omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M). \tag{S-C}$$

Under the assumption (S-C), we define the self-adjoint operator $\overrightarrow{\Delta} = \nabla^* \nabla + R_+ - R_-$ on $L^2(\Lambda^1 T^* M)$ as the operator associated with the form

$$\overrightarrow{\mathfrak{a}}(\omega, \eta) = \overrightarrow{\mathfrak{h}}(\omega, \eta) - \int_M < R_-(x)\omega(x), \eta(x) >_x d\mu,$$

$$\mathcal{D}(\overrightarrow{\mathfrak{a}}) = \mathcal{D}(\overrightarrow{\mathfrak{h}}).$$

It is well known by the KLMN theorem (see [44], Theorem 1.19, p.12) that $\overrightarrow{\mathfrak{a}}$ is a closed form, bounded from below. Therefore it has an associated self-adjoint operator which is $H - R_-$.

In order to use the techniques in [2], we need first to prove that the semigroup $(e^{-t\overrightarrow{\Delta}})_{t \geq 0}$ is uniformly bounded on $L^p(\Lambda^1 T^* M)$ for all $p \in (p_0', 2]$.

3.3 L^p theory of the heat semigroup on forms

To study the boundedness of the semigroup $(e^{-t\overrightarrow{\Delta}})_{t \geq 0}$ on $L^p(\Lambda^1 T^* M)$ for $p \neq 2$, we use perturbation arguments as in [42], where Liskevich and Semenov studied semigroups associated with Schrödinger operators with negative potentials. The main result of this section is the following.

Theorem 3.3.1. *Suppose that the assumptions* (D), (G) *and* $(S\text{-}C)$ *are satisfied. Then the operator* $\overrightarrow{\Delta} = \nabla^* \nabla + R_+ - R_-$ *generates a uniformly bounded analytic semigroup on* $L^p(\Lambda^1 T^* M)$ *for all* $p \in (p_0', p_0)$ *where* $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ *if* $D > 2$ *and* $p_0 = +\infty$ *if* $D \leq 2$.

Note that a slightly weaker statement can be found in [13] Theorem 4.1.15.

To prove **Theorem 3.3.1** we proceed in two steps. The first step consists in proving the result for p in the smaller range $[p_1', p_1]$ where $p_1 = \frac{2}{1-\sqrt{1-\epsilon}}$; we do this in **Proposition 3.3.4**, with the help of **Lemma 3.3.2** below. The second step consists in extending this interval using interpolation between the estimates of **Proposition 3.3.7** and **Proposition 3.3.8**.

We begin with the following lemma.

Lemma 3.3.2. *Let* $p \geq 1$. *For any suitable* $\omega \in \Lambda^1 T^* M$ *and for every* $x \in M$

$$< \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x \; \geq \; \frac{4(p-1)}{p^2} < \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x . \tag{3.3}$$

Remark 3.3.3. In the previous statement, "suitable" means that the calculations make sense with such a ω. For instance, a form $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ is suitable.

Proof. To make the calculations simpler, for every $x \in M$, we work in a synchronous frame. That is we choose an orthonormal frame $\{X_i\}_i$ to have the Christoffel symbols $\Gamma_{ij}^k(x) = 0$ at x (see for instance [32] p.93 or [46] p.70,73 for more details). In what follows, we use properties satisfied by the Levi-Civita connection ∇, which can be found in [46] p.64-66.

Considering $\{\theta^i\}_i$ the orthonormal frame of 1-forms dual to $\{X_i\}_i$, we write for a 1-form ω, $\omega(y) = \sum_i f_i(y)\theta^i = \sum_i \omega_i(y)$ for all y in a neighborhood of x. With this choice of local coordinates we have at x, $|\omega(x)|_x = \sqrt{\sum_i f_i(x)^2}$ and $\nabla\theta^i = 0$ for all i. Then, when $\omega(x) \neq 0$, we obtain

$$\nabla(|\omega|)(x) = \frac{\sum_i f_i(x) df_i(x)}{|\omega(x)|_x}. \tag{3.4}$$

We recall that we have an inner product in each tensor product $T_x^*M \otimes T_x^*M$ satisfying

$$< \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) >_x = < \omega_1(x), \omega_2(x) >_x < \eta_1(x), \eta_2(x) >_x, \tag{3.5}$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$. In particular for all $\omega, \eta \in \Lambda^1 T^*M$ and $x \in M$

$$|\omega(x) \otimes \eta(x)|_x = |\omega(x)|_x |\eta(x)|_x. \tag{3.6}$$

To avoid dividing by 0, one can replace $|\omega(x)|_x$ by $|\omega(x)|_{x,\epsilon} := \sqrt{\sum_i f_i(x)^2 + \epsilon}$ for some $\epsilon > 0$, make the calculations and let ϵ tend to 0. For simplicity, we ignore this step and make the calculations formally.

We first deal with the RHS of (3.3). Using (3.4) and (3.6), we have

$$\begin{aligned}
&< \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x \\
&= \left| |\omega(x)|_x^{\frac{p}{2}-1} \nabla\omega(x) + (\frac{p}{2} - 1)|\omega(x)|_x^{\frac{p}{2}-3} (\sum_i f_i(x) df_i(x)) \otimes \omega(x) \right|_x^2 \\
&= |\omega(x)|_x^{p-2} |\nabla\omega(x)|_x^2 + (\frac{p}{2} - 1)^2 |\omega(x)|_x^{p-6} |\sum_i f_i(x) df_i(x)|_x^2 |\omega(x)|_x^2 \\
&\quad + (p-2)|\omega(x)|_x^{p-4} < \nabla\omega(x), (\sum_i f_i(x) df_i(x)) \otimes \omega(x) >_x .
\end{aligned}$$

Now noticing that $(\theta^i)_i$ is an orthonormal basis of T_x^*M and using (3.5) yield

$$< \nabla\omega(x), (\sum_i f_i(x) df_i(x)) \otimes \omega(x) >_x = < \sum_j df_j(x) \otimes \theta^j, (\sum_i f_i(x) df_i(x)) \otimes \omega(x) >_x$$

$$= \sum_{i,k} f_i(x) f_k(x) < df_i(x), df_k(x) >_x$$
$$= |\sum_i f_i(x) df_i(x)|_x^2.$$

Then we obtain

$$< \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x$$
$$= |\omega(x)|_x^{p-2} |\nabla\omega(x)|_x^2 + (\frac{p^2}{4} - 1)|\omega(x)|_x^{p-4} |\sum_i f_i(x) df_i(x)|_x^2.$$

Using the equality $|\nabla\omega(x)|_x = \sum_i |df_i(x)|_x^2$ at x, a simple calculation gives for all i

$$|\sum_i f_i(x) df_i(x)|_x^2 = \sum_i f_i(x)^2 |df_i(x)|_x^2 + 2\sum_{i<j} f_i(x) f_j(x) < df_i(x), df_j(x) >_x$$
$$= |\omega(x)|_x^2 |\nabla\omega(x)|_x^2 - \sum_i \sum_{j\neq i} f_j(x)^2 |df_i(x)|_x^2 + 2\sum_{i<j} f_i(x) f_j(x) < df_i(x), df_j(x) >_x .$$

Thus for all i

$$|\sum_i f_i(x) df_i(x)|_x^2 = |\omega(x)|_x^2 |\nabla\omega(x)|_x^2 - \sum_{i<j} |f_i(x) df_j(x) - f_j(x) df_i(x)|_x^2. \tag{3.7}$$

Finally we obtain

$$< \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x$$
$$= \frac{p^2}{4} |\omega(x)|_x^2 |\nabla\omega(x)|_x^2 - (\frac{p^2}{4} - 1)|\omega(x)|_x^{p-4} \sum_{i<j} |f_i(x) df_j(x) - f_j(x) df_i(x)|_x^2.$$

Let us deal with the LHS of (3.3) now. We write

$$< \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x = \sum_i < \nabla(\omega_i|\omega|^{p-2})(x), \nabla\omega(x) >_x .$$

Using again (3.5), we observe that for all i, j with $i \neq j$, $< \nabla\omega_i(x), \nabla\omega_j(x) >_x = 0$. Thus, using (3.4), we obtain that for all i

$$< \nabla(\omega_i|\omega|^{p-2})(x), \nabla\omega(x) >_x$$
$$= |\omega(x)|_x^{p-2} |\nabla\omega_i(x)|_x^2 + (p-2)|\omega(x)|_x^{p-4} \sum_j f_j(x) < df_j(x) \otimes \omega_i(x), \nabla\omega(x) >_x .$$

From (3.5) again, we deduce that for all i, j

$$< df_j(x) \otimes \omega_i(x), \nabla\omega(x) >_x = f_i(x) < df_j(x) \otimes \theta^i, \sum_k df_k(x) \otimes \theta^k >_x$$

$$= f_i(x) < df_i(x), df_j(x) >_x .$$

Hence for all i

$$\begin{aligned}
&< \nabla(\omega_i|\omega|^{p-2})(x), \nabla\omega(x) >_x \\
&= |\omega(x)|_x^{p-2}|\nabla\omega_i(x)|_x^2 + (p-2)|\omega(x)|_x^{p-4}\sum_j f_i(x)f_j(x) < df_i(x), df_j(x) >_x .
\end{aligned}$$

As we did before to obtain (3.7), we find

$$\begin{aligned}
&< \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x \\
&= \sum_i < \nabla(\omega_i|\omega|^{p-2})(x), \nabla\omega(x) >_x \\
&= (p-1)|\omega(x)|_x^{p-2}|\nabla\omega(x)|_x^2 - (p-2)|\omega(x)|_x^{p-4}\sum_{i<j}|f_i(x)df_j(x) - f_j(x)df_i(x)|_x^2.
\end{aligned}$$

To conclude we calculate

$$\begin{aligned}
&\frac{1}{p-1} < \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x - \frac{4}{p^2} < \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x \\
&= \left(\frac{4}{p^2}(\frac{p^2}{4} - 1) - \frac{p-2}{p-1}\right)|\omega(x)|_x^{p-4}\sum_{i<j}|f_i(x)df_j(x) - f_j(x)df_i(x)|_x^2 \\
&= \frac{(p-2)^2}{(p-1)p^2}|\omega(x)|_x^{p-4}\sum_{i<j}|f_i(x)df_j(x) - f_j(x)df_i(x)|_x^2 \\
&\geq 0.
\end{aligned}$$

This proves the lemma.

□

We are now able to prove that the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is uniformly bounded on $L^p(\Lambda^1T^*M)$ for some $p \neq 2$ under the assumption (S-C).

Proposition 3.3.4. *Suppose that the negative part R_- of the Ricci curvature satisfies the assumption (S-C). Then the operator $\overrightarrow{\Delta}$ generates a $\mathcal{C}^0$-semigroup of contractions on $L^p(\Lambda^1T^*M)$ for all $p \in [p_1', p_1]$ where $p_1 = \frac{2}{1-\sqrt{1-\epsilon}}$.*

Proof. We consider $\eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$ and set $\omega_t = e^{-t\overrightarrow{\Delta}}\eta$ for all $t \geq 0$. Taking the inner product of both sides of the equation $-\frac{d}{dt}\omega_t = \overrightarrow{\Delta}\omega_t$ with $|\omega_t|^{p-2}\omega_t$ and integrating over M yield

$$-\frac{1}{p}\frac{d}{dt}\|\omega_t\|_p^p = (\overrightarrow{\Delta}\omega_t, |\omega_t|^{p-2}\omega_t)$$

$$= \int_M < \nabla\omega_t(x), \nabla(|\omega_t|^{p-2}\omega_t)(x) >_x d\mu + \left((R_+ - R_-)\omega_t, |\omega_t|^{p-2}\omega_t\right).$$

Since we have by linearity of $R_+(x)$ and $R_-(x)$

$$\left((R_+ - R_-)\omega_t, |\omega_t|^{p-2}\omega\right) = \left((R_+ - R_-)(|\omega_t|^{\frac{p}{2}-1}\omega_t), |\omega_t|^{\frac{p}{2}-1}\omega_t\right),$$

the previous lemma and the assumption (S-C) yield

$$-\frac{1}{p}\frac{d}{dt}\|\omega_t\|_p^p \geq \left(\frac{4(p-1)}{p^2} - \varepsilon\right)\|H^{\frac{1}{2}}(|\omega_t|^{\frac{p}{2}-1}\omega_t)\|_2^2.$$

Then for all $p \in [\frac{2}{1+\sqrt{1-\varepsilon}}, \frac{2}{1-\sqrt{1-\epsilon}}]$

$$-\frac{1}{p}\frac{d}{dt}\|\omega_t\|_p^p \geq 0.$$

Therefore $\|\omega_t\|_p \leq \|\omega_0\|_p$, that is,

$$\|e^{-t\overrightarrow{\Delta}}\eta\|_p \leq \|\eta\|_p, \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M),$$

and we conclude by a usual density argument. □

Actually, as in [42] and [2], we can obtain a better interval than $[p_1', p_1]$ by interpolation arguments and prove **Theorem 3.3.1** . The ideas of this proof are the same as in [2]. However we give some details which we adapt to our setting.

Lemma 3.3.5. *Let q be such that $2 < q \leq \infty$ and $\frac{q-2}{q}D < 2$. Then for all $x \in M$, $t > 0$ and $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{a}})$*

$$\|\chi_{B(x,\sqrt{t})}\omega\|_q \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{2}-\frac{1}{q}}}\left(\|\omega\|_2 + \sqrt{t}\|\overrightarrow{\Delta}^{\frac{1}{2}}\omega\|_2\right).$$

Proof. We recall that H denotes the operator $\nabla^*\nabla + R_+$ and that we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$ for any $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ (see [9] p.171,172). Since we assume (G), the heat kernel $p_t^H(x,y)$ associated to the semigroup $(e^{-tH})_{t\geq 0}$ satisfies a Gaussian upper bound

$$\|p_t^H(x,y)\| \leq \frac{C}{v(x,\sqrt{t})}\exp(-c\frac{\rho^2(x,y)}{t}), \forall t > 0, \forall x, y \in M. \tag{3.8}$$

From (3.8) and the volume doubling property (D), it is not difficult to show that for all $x \in M$ and $0 < s \leq t$

$$\|\chi_{B(x,\sqrt{t})}e^{-sH}\|_{2-\infty} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{2}}}\left(\frac{t}{s}\right)^{\frac{D}{4}}. \tag{3.9}$$

Indeed for $x \in M$, $y \in B(x, \sqrt{t})$ and $0 < s \leq t$, the inclusion of balls

$$B(x, \sqrt{t}) \subset B(y, \sqrt{t} + \rho(x, y)) \subset B(y, 2\sqrt{t})$$

and the volume doubling property yield

$$v(x, \sqrt{t}) \leq C \left(\frac{t}{s}\right)^{\frac{D}{2}} v(y, \sqrt{s}). \tag{3.10}$$

In addition (3.8) implies that for all $x \in M$, $y \in B(x, \sqrt{t})$, $\omega \in L^2(\Lambda^1 T^*M)$ and $0 < s \leq t$

$$|\chi_{B(x,\sqrt{t})}(y) e^{-sH} \omega(y)| \leq \int_M \frac{C}{v(y, \sqrt{s})} \exp(-c\frac{\rho^2(y, z)}{s}) |\omega(z)|_z d\mu(z).$$

Writing $v(y, \sqrt{s}) = v(y, \sqrt{s})^{\frac{1}{2}} v(y, \sqrt{s})^{\frac{1}{2}}$, then using (3.10) and the Hölder inequality, leads to

$$|\chi_{B(x,\sqrt{t})}(y) e^{-sH} \omega(y)| \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}}} \left(\frac{t}{s}\right)^{\frac{D}{4}} \left(\int_M \frac{\exp(-2c\frac{\rho^2(y,z)}{s})}{v(y, \sqrt{s})} d\mu(z)\right)^{\frac{1}{2}} \|\omega\|_2. \tag{3.11}$$

We use a standard decomposition of M into annuli to obtain

$$\begin{aligned}\int_M \exp(-2c\frac{\rho^2(y, z)}{s}) d\mu(z) &\leq \sum_{k=0}^{\infty} \int_{k\sqrt{s} \leq \rho(y,z) \leq (k+1)\sqrt{s}} \exp(-2ck^2) d\mu(z) \\ &\leq \sum_{k=0}^{\infty} \exp(-2ck^2) v(y, (k+1)\sqrt{s}).\end{aligned}$$

Then the volume doubling property (D) implies

$$\int_M \exp(-2c\frac{\rho^2(y, z)}{s}) d\mu(z) \leq C v(y, \sqrt{s}). \tag{3.12}$$

We deduce (3.9) from (3.11) and (3.12).
Now since the semigroup $(e^{-tH})_{t \geq 0}$ is bounded on $L^2(\Lambda^1 T^*M)$, it follows by interpolation that

$$\|\chi_{B(x,\sqrt{t})} e^{-sH}\|_{2-q} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2} - \frac{1}{q}}} \left(\frac{t}{s}\right)^{\frac{D}{2}(\frac{1}{2} - \frac{1}{q})}, \tag{3.13}$$

for all $2 < q \leq \infty$. Note that since the semigroup $(e^{-tH})_{t \geq 0}$ is analytic on $L^2(\Lambda^1 T^*M)$, we have for all $\omega \in L^2(\Lambda^1 T^*M)$ and all $s \geq 0$

$$\|H^{\frac{1}{2}} e^{-sH} \omega\|_2 \leq \frac{C}{\sqrt{s}} \|\omega\|_2. \tag{3.14}$$

Then writing for all $\omega \in \mathcal{D}(\vec{\mathrm{d}})$

$$\omega = e^{-tH}\omega + \int_0^t He^{-sH}\omega\, ds = e^{-tH}\omega + \int_0^t e^{-\frac{s}{2}H}H^{\frac{1}{2}}e^{-\frac{s}{2}H}H^{\frac{1}{2}}\omega\, ds,$$

and using (3.13) and (3.14), we obtain

$$\|\chi_{B(x,\sqrt{t})}\omega\|_q \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{2}-\frac{1}{q}}}\left(\|\omega\|_2 + t^{\frac{D}{2}(\frac{1}{2}-\frac{1}{q})}\|H^{\frac{1}{2}}\omega\|_2 \int_0^t s^{-\frac{1}{2}-\frac{D}{2}(\frac{1}{2}-\frac{1}{q})}ds\right).$$

The convergence of the last integral is ensured for q such that $\frac{q-2}{q}D < 2$ and we then have for such q

$$\|\chi_{B(x,\sqrt{t})}\omega\|_q \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{2}-\frac{1}{q}}}\left(\|\omega\|_2 + \sqrt{t}\|H^{\frac{1}{2}}\omega\|_2\right). \tag{3.15}$$

To conclude the proof, we need to have the estimate (3.15) with the operator $\vec{\Delta}$ instead of H. This is a consequence of the assumption (S-C) since we have for all $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, $\|H^{\frac{1}{2}}\omega\|_2^2 \leq \frac{1}{1-\epsilon}\|\vec{\Delta}^{\frac{1}{2}}\omega\|_2^2$. □

Remark 3.3.6. Lemma 3.3.5 also follows from [13], Proposition 2.3.1 since the heat kernel of H satisfies a Gaussian estimate.

A key result to obtain **Theorem 3.3.1** is the following proposition.

Proposition 3.3.7. *We consider $2 \leq p < p_1$ and q such that $1 \leq q \leq \infty$ and $\frac{q-1}{q}D < 2$. Then for all $x \in M$ and $t > 0$*

$$\|\chi_{B(x,\sqrt{t})}e^{-s\vec{\Delta}}\|_{p-pq} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{p}-\frac{1}{pq}}}\left(\max\left(1, \sqrt{\frac{t}{s}}\right)\right)^{\frac{2}{p}}.$$

Proof. Combining **Lemma 3.3.5**, **Proposition 3.3.4** and following the proof of Proposition 2.2 from [2] lead to the desired result. □

Following the ideas in [2], the last property we need to check is that the semigroup $(e^{-t\vec{\Delta}})_{t\geq 0}$ satisfies the Davies-Gaffney estimates (also called L^2-L^2 off-diagonal estimates in [2]). This is the purpose of the next proposition. Its proof is based on the well-known Davies's perturbation method. Another proof can be found in [48], Theorem 6.

Proposition 3.3.8. *Let E, F be two closed subsets of M. For any $\eta \in L^2(\Lambda^1 T^*M)$ with support in E*

$$\|e^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq e^{-\frac{\rho^2(E,F)}{2t}}\|\eta\|_2.$$

Proof. We choose a constant $\alpha > 0$ and a bounded Lipschitz function ϕ such that $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$. We define the operator $\overrightarrow{\Delta}_\alpha = e^{\alpha\phi}\overrightarrow{\Delta}e^{-\alpha\phi}$ with the sesquilinear form

$$\overrightarrow{a_\alpha}(u,v) = \overrightarrow{a}(e^{-\alpha\phi}u, e^{\alpha\phi}v),\ \mathcal{D}(\overrightarrow{a_\alpha}) = \mathcal{D}(\overrightarrow{a}).$$

Note that since ϕ is bounded then $e^{\pm\alpha\phi}u \in \mathcal{D}(\overrightarrow{a})$ for all $u \in \mathcal{D}(\overrightarrow{a})$.
For $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{a}})$, we have

$$\begin{aligned}
&\Big((\overrightarrow{\Delta}_\alpha + \alpha^2)\omega, \omega\Big)\\
&= \int_M < \nabla(e^{-\alpha\phi}\omega)(x), \nabla(e^{\alpha\phi}\omega)(x) >_x d\mu + ((R_+ - R_-)\omega,\omega) + \alpha^2\|\omega\|_2^2\\
&= \int_M < e^{-\alpha\phi(x)}\nabla\omega(x) - \alpha e^{-\alpha\phi(x)}\nabla\phi(x)\otimes\omega(x),\\
&\qquad\qquad\qquad e^{\alpha\phi(x)}\nabla\omega(x) + \alpha e^{\alpha\phi(x)}\nabla\phi(x)\otimes\omega(x) >_x d\mu\\
&\quad + ((R_+ - R_-)\omega,\omega) + \alpha^2\|\omega\|_2^2\\
&= \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2\int_M |\nabla\phi(x)|_x^2|\omega(x)|_x^2 d\mu - (R_-\omega,\omega) + \alpha^2\|\omega\|_x^2\\
&\geq 0.
\end{aligned}$$

The last inequality follows from the fact that the operator $\overrightarrow{\Delta}$ is non-negative and $|\nabla\phi(x)| \leq 1$ for almost every $x \in M$. As a consequence, the operator $\overrightarrow{\Delta}_\alpha + \alpha^2$ is positive and self-adjoint on $L^2(\Lambda^1T^*M)$ and then $-(\overrightarrow{\Delta}_\alpha + \alpha^2)$ generates a $\mathcal{C}^0$-semigroup of contractions on $L^2(\Lambda^1T^*M)$. Therefore for all $\eta \in L^2(\Lambda^1T^*M)$

$$\|e^{-t\overrightarrow{\Delta}_\alpha}\eta\|_2 \leq e^{t\alpha^2}\|\eta\|_2.$$

Now we consider E and F two closed subsets of M, $\eta \in L^2(\Lambda^1T^*M)$ with support in E and $\phi_k(x) := \min(\rho(x,E),k)$ for $k \in \mathbb{N}$. Since $e^{\alpha\phi_k}\eta = \eta$, we have $e^{-t\overrightarrow{\Delta}}\eta = e^{-\alpha\phi_k}e^{-t\overrightarrow{\Delta}_\alpha}\eta$. Thus we obtain

$$\|e^{-t\overrightarrow{\Delta}}\eta\|_{L^2(F)} \leq e^{-\alpha\min(\rho(E,F),k)}e^{t\alpha^2}\|\eta\|_2.$$

To end the proof, let k tends to infinity and set $\alpha = \frac{\rho(E,F)}{2t}$.

□

Finally we give the proof of **Theorem 3.3.1**.

Proof of Theorem 3.3.1. For $x \in M$, $t \geq 0$ and $k \in \mathbb{N}$, we denote by $A(x,\sqrt{t},k)$ the annulus $B(x,(k+1)\sqrt{t}) \setminus B(x,k\sqrt{t})$. Noticing that

$$\|\chi_{B(x,\sqrt{t})}e^{-t\overrightarrow{\Delta}}\chi_{A(x,\sqrt{t},k)}\|_{p-pq} \leq \|\chi_{B(x,\sqrt{t})}e^{-t\overrightarrow{\Delta}}\|_{p-pq},$$

and using **Proposition 3.3.7**, we obtain the estimate

$$\|\chi_{B(x,\sqrt{t})}e^{-t\overrightarrow{\Delta}}\chi_{A(x,\sqrt{t},k)}\|_{p-pq} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{p}-\frac{1}{pq}}}, \tag{3.16}$$

for all $p \in [2, p_1)$ and q such that $1 \leq q \leq \infty$ and $\frac{q-1}{q}D < 2$. Interpolating (3.16) with the Davies-Gaffney estimate of **Proposition 3.3.8** yields

$$\|\chi_{B(x,\sqrt{t})}e^{-t\overrightarrow{\Delta}}\chi_{A(x,\sqrt{t},k)}\|_{r-s} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{r}-\frac{1}{s}}}e^{-ck^2},$$

for all $r \in [2, p_1)$ and all $s \in (2, p_1 q_0)$ where $q_0 = +\infty$ if $D \leq 2$ and $q_0 = \frac{D}{D-2}$ if $D > 2$. Since the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is analytic on $L^2(\Lambda^1 T^*M)$ and uniformly bounded on $L^p(\Lambda^1 T^*M)$ for all $p \in [p_1', p_1]$, Proposition 3.12 in [44] ensures that it is analytic on $L^p(\Lambda^1 T^*M)$ for all $p \in (p_1', p_1)$. Therefore applying [10] Theorem 1.1, we deduce that $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is bounded analytic on $L^p(\Lambda^1 T^*M)$ for all $p \in [2, p_1 q_0) = [2, p_0)$. The case $p \in (p_0', 2]$ is obtained by a usual duality argument. □

3.4 Proof of Theorem 3.1.1

We start with the following L^p-L^q off-diagonal estimates for the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$, which are consequences of the results of the previous section.

Theorem 3.4.1. *Suppose that (D), (G) and $(S\text{-}C)$ are satisfied. Then for all $r, t > 0$, $x, y \in M$ and all $p \in (p_0', p_0)$, $q \in [p, p_0)$*

(i) $$\|\chi_{B(x,r)}e^{-t\overrightarrow{\Delta}}\chi_{B(y,r)}\|_{p-q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left(\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right)^{\beta} e^{-c\frac{\rho^2(B(x,r),B(y,r))}{t}},$$

(ii) $$\|\chi_{C_j(x,r)}e^{-t\overrightarrow{\Delta}}\chi_{B(x,r)}\|_{p-q} \leq \frac{Ce^{-c\frac{4^j r^2}{t}}}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left(\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r})\right)^{\beta},$$

where $C_j(x,r) = B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ and $\beta \geq 0$ depends on p and q.

Proof. We first treat the case $p \geq 2$.
We recall that from **Proposition 3.3.8**, we have for E and F two closed subsets of M

$$\|\chi_F e^{-t\overrightarrow{\Delta}}\chi_E\|_{2-2} \leq e^{-\frac{\rho^2(E,F)}{2t}}, \tag{3.17}$$

and from **Theorem 3.3.1**, we have for all $p \in (p_0', p_0)$

$$\|e^{-t\overrightarrow{\Delta}}\|_{p-p} \leq C. \tag{3.18}$$

Using the Riesz-Thorin interpolation theorem from (3.17) and (3.18) implies the L^p-L^p off-diagonal estimate

$$\|\chi_F e^{-t\overrightarrow{\Delta}}\chi_E\|_{p-p} \leq Ce^{-c\frac{\rho^2(E,F)}{t}}, \tag{3.19}$$

for all $t \geq 0$ and $p \in (p_0', p_0)$. Taking $p \in [2, p_1)$ and using interpolation from (3.19) and **Proposition 3.3.7** yield

$$\|\chi_{B(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left[\max(1, \frac{r}{\sqrt{t}})\right]^{\beta} e^{-c\frac{\rho^2(B(x,r),B(y,r))}{t}},$$

for $p \in [2, p_1)$ and $u \in [1, \infty)$ if $D \leq 2$ or $u \in [1, \frac{D}{D-2})$ if $D > 2$. Here β is a non-negative constant depending on p and u.
If $D \leq 2$, we have the L^2-L^q off-diagonal estimate for all $q \in [2, +\infty)$.
If $D > 2$, we can deduce, by a composition argument, L^2-L^q off-diagonal estimates for $q \in [2, p_0)$ from L^2-L^p and L^p-L^{pu} off-diagonal estimates with $p \in [2, p_1)$ and $u \in [1, \frac{D}{D-2})$. More precisely, we obtain

$$\|\chi_{B(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left[\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right]^{\beta} e^{-c\frac{\rho^2(B(x,r),B(y,r))}{t}},$$

for all $2 \leq p \leq q < p_0$.
The case $p_0' < p \leq q \leq 2$ is obtained by duality and composition arguments. More precisely, we obtain

$$\|\chi_{B(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left[\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right]^{\beta} e^{-c\frac{\rho^2(B(x,r),B(y,r))}{t}},$$

for all $p_0' < p \leq q < p_0$, which is (i). The reader can find more details in [2] Theorem 2.6.

Now we prove (ii). Writing

$$\chi_{C_j(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(x,r)} = \chi_{C_j(x,r)}\chi_{B(x,2^{j+1}r)} e^{-t\overrightarrow{\Delta}}\chi_{B(x,2^{j+1}r)}\chi_{B(x,r)},$$

it is obvious that

$$\|\chi_{C_j(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(x,r)}\|_{p-q} \leq \|\chi_{B(x,2^{j+1}r)} e^{-t\overrightarrow{\Delta}}\chi_{B(x,2^{j+1}r)}\|_{p-q}. \tag{3.20}$$

Then (i) implies

$$\|\chi_{C_j(x,r)} e^{-t\overrightarrow{\Delta}}\chi_{B(x,r)}\|_{p-q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}}\left[\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r})\right]^{\beta}. \tag{3.21}$$

Using interpolation from (3.19) and (3.21), we deduce that

$$\|\chi_{C_j(x,r)} e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r})\right]^{\beta} e^{-c\frac{\rho^2(C_j(x,r),B(x,r))}{t}},$$

and (ii) follows.

$\square$

In the sequel we prove that the operators $d^* e^{-t\vec{\Delta}}$ and $d\, e^{-t\vec{\Delta}}$ satisfy L^p-L^2 off-diagonal estimates for all $p \in (p_0', 2]$. We need the following lemma.

Lemma 3.4.2. *For any suitable ω and for every $x \in M$*

(i) $|d\omega(x)|_x \leq 2|\nabla\omega(x)|_x$,

(ii) $|d^*\omega(x)|_x \leq \sqrt{N}|\nabla\omega(x)|_x$.

Proof. As we did in the proof of **Lemma 3.3.2**, for every $x \in M$, we work in a synchronous frame to have an orthonormal basis $(\theta^i)_i$ of T_x^*M such that $\nabla\theta^i = 0$ at x. We recall that we have an inner product in each tensor product $T_x^*M \otimes T_x^*M$ satisfying

$$< \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) >_x = < \omega_1(x), \omega_2(x) >_x < \eta_1(x), \eta_2(x) >_x, \tag{3.22}$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$.
If $\omega(x) = f(x)\theta^i$ for a certain i, using (3.22), we have

$$\begin{aligned}
|d\omega(x)|_x^2 &= |df(x) \wedge \theta^i|_x^2 = |\sum_{j=1}^{n} \partial_j f(x)\theta^j \wedge \theta^i|_x^2 \\
&= \sum_{j,k} \partial_j f(x)\partial_k f(x) < \theta^j \otimes \theta^i - \theta^i \otimes \theta^j, \theta^k \otimes \theta^i - \theta^i \otimes \theta^k >_x \\
&= 2\sum_j (\partial_j f(x))^2 - 2(\partial_i f(x))^2.
\end{aligned}$$

Since $\sum_j (\partial_j f(x))^2 = |df(x)|_x^2$ at x, we obtain for $\omega(x) = f(x)\theta^i$

$$|d\omega(x)|_x^2 = 2(|df(x)|_x^2 - (\partial_i f(x))^2). \tag{3.23}$$

Now taking $\eta(x) = g(x)\theta^j$ for $j \neq i$, we have

$$< d\omega(x), d\eta(x) >_x = \sum_{k,l} \partial_k f(x)\partial_l g(x) < \theta^k \otimes \theta^i - \theta^i \otimes \theta^k, \theta^l \otimes \theta^j - \theta^j \otimes \theta^l >_x,$$

which, by (3.22), yields

$$< d\omega(x), d\eta(x) >_x = \; -2\partial_j f(x)\partial_i g(x). \tag{3.24}$$

Thus, in the general case, writing $\omega(x) = \sum_i f_i(x)\theta^i = \sum_i \omega_i(x)$ and using (3.23) and (3.24), we obtain

$$\begin{aligned}
|d\omega(x)|_x^2 &= \sum_i |d\omega_i(x)|_x^2 + \sum_{i\neq j} < d\omega_i(x), d\omega_j(x) >_x \\
&= 2\sum_i (|df_i(x)|_x^2 - (\partial_i f_i(x))^2) - 2\sum_{i\neq j} \partial_j f_i(x)\partial_i f_j(x) \\
&= 2|\nabla\omega(x)|_x^2 - \sum_{i,j}(\partial_j f_i(x) + \partial_i f_j(x))^2 + 2\sum_{i,j}(\partial_i f_j(x))^2 \\
&= 2|\nabla\omega(x)|_x^2 - \sum_{i,j}(\partial_j f_i(x) + \partial_i f_j(x))^2 + 2|\nabla\omega(x)|_x^2 \\
&\leq 4|\nabla\omega(x)|_x^2,
\end{aligned}$$

which gives $i)$. To prove $ii)$, we notice that $d^*\omega(x) = -\sum_i \partial_i f_i(x)$ at x (see for instance [46] p.19). Hence using the Cauchy-Schwarz inequality and the previous calculations, we have

$$\begin{aligned}
|d^*\omega(x)|_x^2 \; &\leq N\sum_i (\partial_i f_i(x))^2 \\
&= N\left(|\nabla\omega(x)|_x^2 - \frac{1}{4}|d\omega(x)|_x^2 - \frac{1}{4}\sum_{i\neq j}(\partial_j f_i(x) + \partial_i f_j(x))^2\right) \\
&\leq N|\nabla\omega(x)|_x^2.
\end{aligned}$$

□

We will need the following L^2-L^2 off-diagonal estimate.

Proposition 3.4.3. *Assume that (S-C) is satisfied. Let E, F be two closed subsets of M. For any $\eta \in L^2(\Lambda^1 T^*M)$ with support in E we have*

$$\|\nabla e^{-t\overrightarrow{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2.$$

Proof. As in the proof of **Proposition 3.3.8**, we set $\overrightarrow{\Delta}_\alpha = e^{\alpha\phi}\overrightarrow{\Delta}e^{-\alpha\phi}$ where $\alpha > 0$ is a constant and ϕ is a bounded Lipschitz function such that $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$. Using the assumption (S-C), we obtain for $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{a}})$

$$\left((\overrightarrow{\Delta}_\alpha + \alpha^2)\omega, \omega\right) = \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2 \int_M |\nabla\phi(x)|_x^2 |\omega(x)|_x^2 d\mu - (R_-\omega, \omega) + \alpha^2\|\omega\|_x^2$$

$$\geq \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2\|\omega\|_2^2 - (R_-\omega,\omega) + \alpha^2\|\omega\|_2^2$$
$$\geq (1-\epsilon)\|H^{\frac{1}{2}}\omega\|_2^2$$
$$\geq (1-\epsilon)\|\nabla\omega\|_2^2.$$

We recall that from the proof of **Proposition 3.3.8**, one has for $\eta \in L^2(\Lambda^1 T^*M)$

$$\|e^{-t\overrightarrow{\Delta}_\alpha}\eta\|_2 \leq e^{t\alpha^2}\|\eta\|_2. \tag{3.25}$$

Lemma 3.4.4 below ensures that the operator $\overrightarrow{\Delta}_\alpha + 2\alpha^2$ is sectorial. As a consequence the semigroup $(e^{-z(\overrightarrow{\Delta}_\alpha+2\alpha^2)})_{t\geq 0}$ is analytic on the sector $\Sigma = \{z \in \mathbb{C}, z \neq 0, |arg(z)| \leq \frac{\pi}{2} - Arctan(\gamma)\}$ (where γ is the constant appearing in (3.29) below) and $\|e^{-z(\overrightarrow{\Delta}_\alpha+2\alpha^2)}\|_{2,2} \leq 1$ for all $z \in \Sigma$ (see [44] Theorem 1.53, 1.54). A classical argument using the Cauchy formula implies that for all $t \geq 0$

$$\|(\overrightarrow{\Delta}_\alpha + 2\alpha^2)e^{-t(\overrightarrow{\Delta}_\alpha+2\alpha^2)}\|_{2-2} \leq \frac{C}{t}, \tag{3.26}$$

where the constant C does not depend on α. We notice that for every $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{a}})$

$$\Big((\overrightarrow{\Delta}_\alpha + 2\alpha^2)\omega,\omega\Big) \;\geq\; \Big((\overrightarrow{\Delta}_\alpha + \alpha^2)\omega,\omega\Big) \;\geq\; (1-\epsilon)\|\nabla\omega\|_2^2. \tag{3.27}$$

Then setting $\omega = e^{-t(\overrightarrow{\Delta}_\alpha+2\alpha^2)}\eta$ for $\eta \in L^2(\Lambda^1 T^*M)$ and $t \geq 0$, we deduce from (3.25), (3.26) and (3.27) that

$$\|\nabla e^{-t(\overrightarrow{\Delta}_\alpha+2\alpha^2)}\eta\|_2 \leq \frac{C}{\sqrt{t}}\|e^{-t(\overrightarrow{\Delta}_\alpha+2\alpha^2)}\eta\|_2 \leq \frac{C}{\sqrt{t}}\|\eta\|_2, \forall t > 0. \tag{3.28}$$

As we did in the proof of **Proposition 3.3.8** let E and F two closed subsets of M, $\eta \in L^2(\Lambda^1 T^*M)$ with support in E and $\phi_k(x) := \min(\rho(x,E),k)$ for $k \in \mathbb{N}$. Since $e^{\alpha\phi_k}\eta = \eta$, we have $e^{-t\overrightarrow{\Delta}}\eta = e^{-\alpha\phi_k}e^{-t\overrightarrow{\Delta}_\alpha}\eta$. Then we obtain

$$\nabla e^{-t\overrightarrow{\Delta}}\eta = -\alpha e^{-\alpha\phi_k}\nabla\phi_k \otimes e^{-t\overrightarrow{\Delta}_\alpha}\eta + e^{-\alpha\phi_k}\nabla e^{-t\overrightarrow{\Delta}_\alpha}\eta.$$

Since $|\nabla\phi_k(x)|_x \leq 1$ for almost every $x \in M$, we deduce from (3.25) and (3.28) that

$$\|\chi_F\nabla e^{-t\overrightarrow{\Delta}}\eta\|_2 \leq \alpha e^{-\alpha\min(\rho(E,F),k)}e^{t\alpha^2}\|\eta\|_2 + \frac{C}{\sqrt{t}}e^{-\alpha\min(\rho(E,F),k)}e^{2t\alpha^2}\|\eta\|_2.$$

Now letting k tends to infinity and setting $\alpha = \frac{\rho(E,F)}{4t}$, we finally obtain

$$\|\chi_F\nabla e^{-t\overrightarrow{\Delta}}\eta\|_2 \leq \frac{C}{\sqrt{t}}(1 + \frac{\rho(E,F)}{4\sqrt{t}})e^{-\frac{\rho^2(E,F)}{8t}}\|\eta\|_2$$
$$\leq \frac{C}{\sqrt{t}}e^{-c\frac{\rho^2(E,F)}{t}}\|\eta\|_2,$$

which is the desired result. □

In the following lemma, we study sectoriality. Then we need to work with complex valued 1-forms. This is achieved as usual by introducing the complex Hilbert spaces $L^2(\Lambda^1 T^*M) \oplus iL^2(\Lambda^1 T^*M)$ and $\mathcal{D}(\overrightarrow{a}) \oplus i\mathcal{D}(\overrightarrow{a})$.

Lemma 3.4.4. *Under the assumption (S-C), the operator $\overrightarrow{\Delta}_\alpha + 2\alpha^2$ is sectorial. That is there exists a constant $\gamma \geq 0$ such that for all $\omega \in \mathcal{D}(\overrightarrow{\Delta}_\alpha + 2\alpha^2)$*

$$|Im((\overrightarrow{\Delta}_\alpha + 2\alpha^2)\omega, \omega)| \leq \gamma\, Re((\overrightarrow{\Delta}_\alpha + 2\alpha^2)\omega, \omega) \tag{3.29}$$

Proof. We consider $\omega \in \mathcal{D}(\overrightarrow{a}) \oplus i\mathcal{D}(\overrightarrow{a})$. Since $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$, we have

$$\begin{aligned}
\overrightarrow{a_\alpha}(\omega, \omega) = &\overrightarrow{a}(\omega, \omega) + \alpha \int_M < \nabla\omega(x), \nabla\phi(x) \otimes \overline{\omega(x)} >_x d\mu \\
&- \alpha \int_M < \nabla\phi(x) \otimes \omega(x), \overline{\nabla\omega(x)} >_x d\mu - \alpha^2 \int_M |\nabla\phi(x)^2|_x |\omega(x)|_x^2 d\mu \\
&\geq \overrightarrow{a}(\omega, \omega) + 2i\alpha Im\left(\int_M < \nabla\phi(x) \otimes \omega(x), \overline{\nabla\omega(x)} >_x d\mu\right) - \alpha^2 \|\omega\|_2^2.
\end{aligned}$$

Therefore we deduce that

$$Re(\overrightarrow{a_\alpha}(\omega, \omega) + 2\alpha^2\|\omega\|_2^2) \geq \overrightarrow{a}(\omega, \omega) \tag{3.30}$$

$$Re(\overrightarrow{a_\alpha}(\omega, \omega) + 2\alpha^2\|\omega\|_2^2) \geq \alpha^2\|\omega\|_2^2. \tag{3.31}$$

Furthermore, the Cauchy-Schwarz inequality and the assumption (S-C) yield

$$\begin{aligned}
|Im(\overrightarrow{a_\alpha}(\omega, \omega) + 2\alpha^2\|\omega\|_2^2)| &= \left|2\alpha Im\left(\int_M < \nabla\phi(x) \otimes \omega(x), \overline{\nabla\omega(x)} >_x d\mu\right)\right| \\
&\leq 2\alpha \int_M |\omega(x)|_x |\nabla\phi(x)|_x |\nabla\omega(x)|_x d\mu \\
&\leq 2\alpha\|\omega\|_2\|\nabla\omega\|_2 \\
&\leq 2\alpha\|\omega\|_2\|H^{\frac{1}{2}}\omega\|_2 \\
&\leq 2\alpha\sqrt{\frac{1}{1-\epsilon}}\|\omega\|_2\, \overrightarrow{a}^{\frac{1}{2}}(\omega, \omega) \\
&\leq \frac{1}{1-\epsilon}\overrightarrow{a}(\omega, \omega) + \alpha^2\|\omega\|_2^2.
\end{aligned}$$

Using (3.30) and (3.31), we deduce that there exists a constant C_ϵ such that

$$|Im(\overrightarrow{a_\alpha}(\omega, \omega) + 2\alpha^2\|\omega\|_2^2)| \leq C_\epsilon Re(\overrightarrow{a_\alpha}(\omega, \omega) + 2\alpha^2\|\omega\|_2^2),$$

which means that $\overrightarrow{\Delta}_\alpha + 2\alpha^2$ is sectorial. (see [44] Proposition 1.27) $\square$

An immediate consequence of **Lemma 3.4.2** and **Proposition 3.4.3** is the following result.

Corollary 3.4.5. *Assume that (S-C) is satisfied. Let E, F be two closed subsets of M. For any $\eta \in L^2(\Lambda^1 T^* M)$ with support in E*

(i) $\|de^{-t\overrightarrow{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2$,

(ii) $\|d^* e^{-t\overrightarrow{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2$.

We are now able to prove L^p-L^2 off-diagonal estimates for the operators $d^* e^{-t\overrightarrow{\Delta}}$ and $d\, e^{-t\overrightarrow{\Delta}}$.

Theorem 3.4.6. *Suppose that (D), (G) and (S-C) are satisfied. Then for all $r, t > 0$, $x, y \in M$ and all $p \in (p_0', 2]$*

$$\|\chi_{C_j(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{C e^{-c\frac{4^j r^2}{t}}}{\sqrt{t}\, v(x,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right)^{\beta} 2^{j\beta}, \tag{3.32}$$

$$\|\chi_{C_j(x,r)} d^* e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{C e^{-c\frac{4^j r^2}{t}}}{\sqrt{t}\, v(x,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right)^{\beta} 2^{j\beta}, \tag{3.33}$$

where $C_j(x,r) = B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ and $\beta \geq 0$ depends on p.

Proof. We only prove (3.32) since (3.33) can be obtained in the same manner. By **Corollary 3.4.5**, we have for all $x, z \in M$ and $r, t \geq 0$

$$\|\chi_{B(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(z,r)}\|_{2-2} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(B(x,r),B(z,r))}{t}}.$$

In addition by **Theorem 3.4.1**, we have for all $y, z \in M$, $r, t \geq 0$ and $p \in (p_0', 2]$

$$\|\chi_{B(z,r)} e^{-t\overrightarrow{\Delta}} \chi_{B(y,r)}\|_{p-2} \leq \frac{C}{v(z,r)^{\frac{1}{p}-\frac{1}{2}}} e^{-c\frac{\rho^2(B(y,r),B(z,r))}{t}}.$$

Then writing $d\, e^{-t\overrightarrow{\Delta}} = d\, e^{-\frac{t}{2}\overrightarrow{\Delta}} e^{-\frac{t}{2}\overrightarrow{\Delta}}$ and using a composition argument, we obtain

$$\|\chi_{B(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(y,r)}\|_{p-2} \leq \frac{C}{\sqrt{t}\, v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})\right)^{\beta} e^{-c\frac{\rho^2(B(x,r),B(y,r))}{t}}. \tag{3.34}$$

For more details on the composition argument see [2] Theorem 3.5.

Writing $\chi_{C_j(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)} = \chi_{C_j(x,r)} \chi_{B(x,2^{j+1}r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,2^{j+1}r)} \chi_{B(x,r)}$, we notice that

$$\|\chi_{C_j(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \|\chi_{B(x,2^{j+1}r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,2^{j+1}r)}\|_{p-2}$$

Then (3.34) yields

$$\begin{aligned}\|\chi_{C_j(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} &\leq \frac{C}{\sqrt{t}\, v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}\right)\right)^{\beta} \\ &\leq \frac{C 2^{j\beta}}{\sqrt{t}\, v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right)\right)^{\beta}.\end{aligned}$$

Using **Corollary 3.4.5**, we have

$$\|\chi_{C_j(x,r)} d\, e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{2-2} \leq \frac{C}{\sqrt{t}} e^{-c\frac{4^j r^2}{t}}. \tag{3.35}$$

Therefore applying the Riesz-Thorin interpolation theorem from (3.34) and (3.35), we deduce the result.

□

A key result to prove the boundedness of the Riesz transforms $d^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ and $d\overrightarrow{\Delta}^{-\frac{1}{2}}$ is a result in [11] which we state as it is formulated in [3], Theorem 2.1.

Theorem 3.4.7. *Let $p \in (1,2]$. Suppose that T is a sublinear operator of strong type $(2,2)$, and let $(A_r)_{r>0}$ be a family of linear operators acting on L^2. Assume that for $j \geq 2$ and every ball $B = B(x,r)$*

$$\left(\frac{1}{v(x,2^{j+1}r)} \int_{C_j(x,r)} |T(I - A_r) f|^2\right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{v(x,r)} \int_B |f|^p\right)^{\frac{1}{p}}, \tag{3.36}$$

and for $j \geq 1$

$$\left(\frac{1}{v(x,2^{j+1}r)} \int_{C_j(x,r)} |A_r f|^2\right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{v(x,r)} \int_B |f|^p\right)^{\frac{1}{p}}, \tag{3.37}$$

for all f supported in B. If $\Sigma := \sum_j g(j) 2^{Dj} < \infty$, then T is of weak type (p,p), with a bound depending only on the strong type $(2,2)$ bound of T, p and Σ.

Finally we prove **Theorem 3.1.1**.

Proof of Theorem 3.1.1. We argue as in [2] Theorem 3.6. We set $T = d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ and consider the operators $A_r = I - (I - e^{-r^2\overrightarrow{\Delta}})^m$ for some sufficiently large integer m. The estimate (3.37) can be obtained using the estimate

$$\|\chi_{C_j(x,r)} e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{Ce^{-c\frac{4^j r^2}{t}}}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}) \right)^{\beta},$$

which we proved in **Theorem 3.4.1** (see [2] Theorem 3.6).

The estimate (3.36) can be obtained using the estimate

$$\|\chi_{C_j(x,r)} d^* e^{-t\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{Ce^{-c\frac{4^j r^2}{t}}}{\sqrt{t}\, v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}) \right)^{\beta} 2^{j\beta},$$

which we proved in **Theorem 3.4.6** (see [2] Theorem 3.6).

The proof is the same for $T = d\overrightarrow{\Delta}^{-\frac{1}{2}}$. □

3.5 Sub-criticality and proof of Theorem 3.1.3

The assumption (S-C) can be understood as a "smallness" condition on the negative part R_- of the Ricci curvature. But since R_- is a geometric component of the manifold M, it would be interesting to have analytic or geometric conditions which lead to this assumption. This is the purpose of this section.

We recall that Devyver [28] studied the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ where M is a complete non-compact Riemannian manifold satisfying a global Sobolev type inequality

$$\|f\|_{\frac{2N}{N-2}} \leq C\|df\|_2, \forall f \in \mathcal{C}_0^\infty(M).$$

Assuming $R_- \in L^{\frac{N}{2}}$, he proved that R_- satisfies the assumption (S-C) if and only if the space

$$Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) := \{\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}) : \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), (\omega, \overrightarrow{\Delta}\eta) = 0\}$$

is trivial. Here $\mathfrak{h}$ denotes the sesquilinear form defined for all $\omega, \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ by

$$\overrightarrow{\mathfrak{h}}(\omega,\eta) = \int_M < \nabla\omega(x), \nabla\eta(x) >_x \, d\mu + \int_M < R_+(x)\omega(x), \eta(x) >_x \, d\mu,$$

$$\text{and } \mathcal{D}(\overrightarrow{\mathfrak{h}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|.\|_{\overrightarrow{\mathfrak{h}}}},$$

where $\|\omega\|_{\overrightarrow{\mathfrak{h}}} = \sqrt{\overrightarrow{\mathfrak{h}}(\omega,\omega) + \|\omega\|_2^2}$. We recall that H denotes its associated operator, that is, $H = \nabla^*\nabla + R_+$.

Assaad and Ouhabaz introduced in [2] the following quantities

$$\alpha_1 = \int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}}, \ \ \alpha_2 = \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}},$$

for some $r_1, r_2 > 2$. We set $\|R_-^{\frac{1}{2}}\|_{vol} := \alpha_1 + \alpha_2$. We are interested in the finiteness of this norm. It is clear that if the volume is polynomial, that is, $c\,r^N \le v(x,r) \le Cr^N$, then $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$ if and only if $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$. The latter condition is usually assumed to study the boundedness of Riesz transforms of Schrödinger operators on L^p for $p > 2$.

We state the main result of this section.

Theorem 3.5.1. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then R_- satisfies (S-C) if and only if $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) = \{0\}$.*

We can observe that this result is similar to the one of Devyver. However, we do not assume any global Sobolev inequality. In this context, with the additional assumption that the balls of great radius has polynomial volume growth, Definition 2.2.2 in [28] allows $R_- \in L^{\frac{N}{2}}$; whereas in **Theorem 3.5.1**, one needs $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$ with the same condition on the volume.
Assuming **Theorem 3.5.1**, we are now able to prove **Theorem 3.1.3**.

Proof of Theorem 3.1.3. According to the commutation formula $\overrightarrow{\Delta} d = d\Delta$, we see that the adjoint operator of $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is exactly $d\Delta^{-\frac{1}{2}}$. Then **Theorem 3.1.3** is an immediate consequence of **Theorem 3.5.1** and **Theorem 3.1.1**. $\square$

Remark 3.5.2. Let us make a comment on the space $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta})$. We consider $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}})$. Since $\overrightarrow{\Delta}$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\Lambda^1 T^*M)$ (see [52] Section 2), the condition

$$(\omega, \overrightarrow{\Delta}\eta) = 0, \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$$

implies

$$(\omega, \overrightarrow{\Delta}\eta) = 0, \forall \eta \in \mathcal{D}(\overrightarrow{\Delta}).$$

Then $\omega \in \mathcal{D}(\overrightarrow{\Delta})$ and $\overrightarrow{\Delta}\omega = 0$. Therefore $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta})$ is the space of harmonic L^2 forms.

The following proposition proves the first part of **Theorem 3.5.1**.

Proposition 3.5.3. *Assume that M satisfies (D), (G) and that R_- satisfies (S-C). Then* $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$.

Proof. Any ω in $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$ satisfies for all $\eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, $(\vec{\Delta}\omega, \eta) = 0$, hence, by a density argument $(\vec{\Delta}\omega, \omega) = 0$. If R_- satisfies (S-C), we have $(H\omega, \omega) \leq \frac{1}{1-\epsilon}(\vec{\Delta}\omega, \omega) = 0$, which yields $\omega \in Ker(H^{\frac{1}{2}})$. According to **Lemma 3.5.4** below, we deduce that $\omega = 0$. Thus $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$. □

The following result is well-known but we have decided to give its proof for the sake of completeness.

Lemma 3.5.4. *Assume that (D) and (G) are satisfied. Then $Ker(H) = \{0\}$.*

Proof. We consider $\omega \in Ker(H)$, that is $\omega \in \mathcal{D}(H)$ and $H\omega = 0$. We then have for all $t \geq 0$

$$e^{-tH}\omega = \omega. \tag{3.38}$$

Noticing that we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$ and using (3.38) and (G), we obtain for all $x \in M$ and $t \geq 0$

$$|\omega(x)|_x \leq \frac{C}{v(x, \sqrt{t})} \int_M \exp(-c\frac{\rho^2(x,y)}{t})|\omega(y)|_y \, d\mu.$$

The Hölder inequality yields

$$|\omega(x)|_x \leq \frac{C}{v(x, \sqrt{t})} \left(\int_M \exp(-2c\frac{\rho^2(x,y)}{t}) d\mu \right)^{\frac{1}{2}} \|\omega\|_2. \tag{3.39}$$

Using (3.12) in (3.39) leads to

$$|\omega(x)|_x \leq \frac{C}{\sqrt{v(x, \sqrt{t})}} \|\omega\|_2. \tag{3.40}$$

Since the manifold M is connected, complete, non-compact and satisfies the volume doubling property (D), it follows from [35] p.412 that there exists a constant $D' > 0$ such that for all $x \in M$ and $0 < r \leq R$

$$\frac{v(x,R)}{v(x,r)} \geq c \left(\frac{R}{r}\right)^{D'}. \tag{3.41}$$

We obtain from (3.40) and (3.41) that for all $t \geq 1$

$$|\omega(x)|_x \leq \frac{C}{t^{\frac{D'}{4}} \sqrt{v(x,1)}} \|\omega\|_2.$$

Letting t tend to infinity, we deduce that for all $x \in M$, $|\omega(x)|_x = 0$ and then that $Ker(H) = \{0\}$.

□

Note that the assumption $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$ is not necessary in the proof of **Proposition 3.5.3** but will be used to prove the converse of **Theorem 3.5.1**.

Before giving the other half of the proof of **Theorem 3.5.1**, we need the following two results.

Lemma 3.5.5. *Assume that (D) and (G) are satisfied. Then there exists a constant $C \geq 0$ such that*

$$\|R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \leq C \|R_-^{\frac{1}{2}}\|_{vol}$$

and

$$\|H^{-\frac{1}{2}} R_-^{\frac{1}{2}}\|_{2-2} \leq C \|R_-^{\frac{1}{2}}\|_{vol}.$$

Proof. Writing $H^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_0^\infty e^{-tH} \frac{dt}{\sqrt{t}}$ and using the Hölder inequality, we obtain

$$\begin{aligned}
&\|R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \\
&\leq C \int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} v(.,\sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-2} \frac{dt}{\sqrt{t}} + C \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} v(.,\sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-2} \frac{dt}{\sqrt{t}} \\
&\leq C \int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \left\| v(.,\sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-\frac{2r_1}{r_1-2}} \frac{dt}{\sqrt{t}} \\
&\qquad + C \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \left\| v(.,\sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-\frac{2r_2}{r_2-2}} \frac{dt}{\sqrt{t}}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\|H^{-\frac{1}{2}} R_-^{\frac{1}{2}}\|_{2-2} \\
&\leq C \int_0^1 \left\| e^{-tH} v(.,\sqrt{t})^{\frac{1}{r_1}} \right\|_{\frac{2r_1}{r_1+2}-2} \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}} \\
&\qquad + C \int_1^\infty \left\| e^{-tH} v(.,\sqrt{t})^{\frac{1}{r_2}} \right\|_{\frac{2r_2}{r_2+2}-2} \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}}.
\end{aligned}$$

The assumptions (D) and (G) allow us to use Proposition 2.9 in [2] for Δ. Then noticing we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$, for all $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M)$ leads to the following estimates

$$\|v(.,\sqrt{t})^{\frac{1}{p}-\frac{1}{q}} e^{-tH}\|_{p-q} \leq C, \ \forall\, 1 < p \leq q < \infty,$$

where C is a non-negative constant depending on p, q, (D) and (G). By duality

$$\|e^{-tH}v(.,\sqrt{t})^{\frac{1}{p}-\frac{1}{q}}\|_{p-q} \leq C, \ \forall\, 1 < p \leq q < \infty.$$

Since for $i = 1, 2$ we have $\frac{1}{r_i} = \frac{1}{2} - \frac{r_i-2}{2r_i}$ and $\frac{1}{r_i} = \frac{r_i+2}{2r_i} - \frac{1}{2}$, we obtain the desired result. $\square$

As a consequence

Corollary 3.5.6. *The L^2-adjoint of the operator $R_-^{\frac{1}{2}}H^{-\frac{1}{2}}$ is $H^{-\frac{1}{2}}R_-^{\frac{1}{2}}$.*

We now follow the ideas of Devyver to prove **Theorem 3.5.1**. We will need to prove that M is non-parabolic in the following sense

Definition 3.5.7. We say that M is a non-parabolic manifold if

$$\forall x \neq y \in M, \ \int_0^\infty p_t(x,y)dt < \infty.$$

This is the purpose of the next lemma.

Lemma 3.5.8. *Assume that $R_- \neq 0$ and that*

$$\int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < \infty.$$

for a certain $r_2 > 2$. Then for all $x_0 \in M$, there exists two positive constants $C = C(x_0, |R_-|)$ and $R_0 = R_0(x_0, |R_-|)$ such that $\int_{R_0^2}^\infty \frac{dt}{v(x_0,\sqrt{t})} \leq C$. In particular, the manifold M is non-parabolic.

Proof. Let $x_0 \in M$. Since $R_- \neq 0$ on M, one can find a sufficiently large $R_0 > 0$ so that $\chi_{\overline{B}(x_0,R_0)}R_- \neq 0$ on M. Hence $\|\chi_{\overline{B}(x_0,R_0)}|R_-|^{\frac{1}{2}}\|_{r_2} \neq 0$.

Notice that $\int_1^\infty \left\| \frac{\chi_{\overline{B}(x_0,R_0)}|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < \infty$ since $\int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < \infty$.

Furthermore, using (D), we have for all $y \in \overline{B}(x_0, R_0)$ and $t \geq R_0^2$

$$v(y,\sqrt{t}) \leq v(x_0, \sqrt{t} + \rho(x_0,y)) \leq \left(1 + \frac{\rho(x_0,y)}{\sqrt{t}}\right)^D v(x_0,\sqrt{t}) \leq 2^D v(x_0,\sqrt{t}).$$

We deduce that

$$\|\chi_{\overline{B}(x_0,R_0)}|R_-|^{\frac{1}{2}}\|_{r_2} \int_{R_0^2}^\infty \frac{1}{v(x_0,\sqrt{t})^{\frac{1}{r_2}}} \frac{dt}{\sqrt{t}} \leq C \int_{R_0^2}^\infty \left\| \frac{\chi_{\overline{B}(x_0,R_0)}|R_-|^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}} < \infty.$$

It follows that $\displaystyle\int_{R_0^2}^{\infty} \frac{1}{v(x_0,\sqrt{t})^{\frac{1}{r_2}}} \frac{dt}{\sqrt{t}} < \infty$. Then, applying the mean value theorem to the bounded function $s \mapsto \displaystyle\int_{R_0^2}^{s} \frac{dt}{\sqrt{t}v(x_0,\sqrt{t})^{\frac{1}{r_2}}}$ between R_0^2 and t gives $\sqrt{t}v(x_0,\sqrt{t})^{\frac{1}{r_2}} \geq Ct$ for all $t \geq R_0^2$. Hence $\displaystyle\int_{R_0^2}^{\infty} \frac{dt}{v(x_0,\sqrt{t})} \leq C \int_{R_0^2}^{\infty} \frac{dt}{t^{\frac{r_2}{2}}} < \infty$ since $r_2 > 2$.

The fact that M is non-parabolic is then an immediate consequence of [35] Corollary 9.9. $\square$

Note that in **Lemma 3.5.8** we assume $R_- \neq 0$. This is not restrictive because if $R_- = 0$, then R_- obviously satisfies (S-C) and **Theorem 3.5.1** is obvious.

Even if the two lemmas below are known, we give their proofs for the sake of completeness. The following lemma is similar to Proposition 4 and Lemma 1 in [28].

Lemma 3.5.9. *Let Λ denote the $self-adjoint$ operator $H^{-\frac{1}{2}}R_-H^{-\frac{1}{2}} = (R_-^{\frac{1}{2}}H^{-\frac{1}{2}})^*(R_-^{\frac{1}{2}}H^{-\frac{1}{2}})$ acting on $L^2(\Lambda^1T^*M)$. Assume that (D) and (G) are satisfied. Then*

(i) the operator $H^{\frac{1}{2}}$ extends uniquely to an isomorphism from $\mathcal{D}(\overrightarrow{\mathfrak{h}})$ to $L^2(\Lambda^1T^*M)$.

(ii) the operator $H^{\frac{1}{2}}$ is an isomorphism from $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta})$ to $Ker_{L^2}(I-\Lambda)$.

Proof. (i) See [28] Proposition 4 and [27] Definition 3.2.

(ii) We consider $\omega \in Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta})$, that is, $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}})$ such that for all $\eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$

$$(\omega, \overrightarrow{\Delta}\eta) = 0.$$

Let $\eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$. We write $\overrightarrow{\Delta}\eta = H^{\frac{1}{2}}(I-\Lambda)H^{\frac{1}{2}}\eta$. Since $\mathcal{D}(\overrightarrow{\mathfrak{h}}) = \mathcal{D}(H^{\frac{1}{2}})$, we may write

$$\omega \in Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) \iff \omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}) \text{ and } \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M), (H^{\frac{1}{2}}\omega, (I-\Lambda)H^{\frac{1}{2}}\eta) = 0.$$

We claim that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1T^*M))$ is dense in $L^2(\Lambda^1T^*M)$. Assuming the claim, we obtain

$$\omega \in Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) \iff \omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}) \text{ and } (H^{\frac{1}{2}}\omega, (I-\Lambda)\eta) = 0, \forall \eta \in L^2(\Lambda^1T^*M).$$

Noticing that $I-\Lambda$ is self-adjoint on $L^2(\Lambda^1T^*M)$, we deduce that

$$\omega \in Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) \iff \omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}) \text{ and } H^{\frac{1}{2}}\omega \in Ker_{L^2}(I-\Lambda),$$

which, combined with (i) give (ii).

Now we prove the claim. We consider $u = H^{\frac{1}{2}}v \in Im(H^{\frac{1}{2}})$ satisfying

$$(u, H^{\frac{1}{2}}w) = 0, \forall w \in \mathcal{C}_0^\infty(\Lambda^1 T^*M).$$

Then for all $w \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, we have $\overrightarrow{\mathfrak{h}}(v, w) = 0$. Therefore $v \in \mathcal{D}(H)$ and $Hv = 0$, that is $v \in Ker(H)$. Since $Ker(H) = \{0\}$ (see **Lemma 3.5.4** above), we obtain $v = 0$ in $\mathcal{D}(H)$ and then $u = 0$ in $Im(H^{\frac{1}{2}})$. This shows that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1 T^*M))$ is dense in $Im(H^{\frac{1}{2}})$. Furthermore, $Im(H^{\frac{1}{2}})$ is dense in $L^2(\Lambda^1 T^*M)$ because $H^{\frac{1}{2}}$ is self-adjoint and $Ker(H^{\frac{1}{2}}) = \{0\}$. Hence we deduce that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1 T^*M))$ is dense in $L^2(\Lambda^1 T^*M)$. $\square$

The following lemma is similar to Proposition 1.4 and Theorem 1.5 in [14].

Lemma 3.5.10. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then Λ is a compact operator on $L^2(\Lambda^1 T^*M)$.*

Proof. It follows from the same proof as in **Lemma 3.5.5**, applied to $\chi_{B(x,r)^C} R_-^{\frac{1}{2}}$ rather than $R_-^{\frac{1}{2}}$, that we have for all $x \in M$ and $r \geq 0$

$$\|\chi_{B(x,r)^C} R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \leq C \|\chi_{B(x,r)^C} R_-^{\frac{1}{2}}\|_{vol},$$

where $B(x,r)^C$ denotes $M \setminus B(x,r)$. In addition the dominated convergence theorem applied twice ensures that for all $x \in M$

$$\lim_{r \to +\infty} \|\chi_{B(x,r)^C} R_-^{\frac{1}{2}}\|_{vol} = 0.$$

Therefore we deduce that

$$\lim_{r \to +\infty} \chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}} = R_-^{\frac{1}{2}} H^{-\frac{1}{2}},$$

where the limit is the operator limit in $\mathcal{L}(L^2(\Lambda^1 T^*M))$.
We recall that the operator limit in the uniform sense of compact operators is compact. Then to prove the lemma, it suffices to show that the operator $\chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^*M)$ for all $x \in M$ and $r \geq 0$. Since R_- is continuous on M, $R_- \in L^\infty_{loc}(M)$ and then there exists $\phi \in \mathcal{C}_0^\infty(M)$ such that $\phi = 1$ on $B(x,r)$, $\phi \leq 1$ on $B(x,r)^C$ and

$$\|\chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}} \omega\|_2 \leq C \|\phi H^{-\frac{1}{2}} \omega\|_2, \forall \omega \in L^2(\Lambda^1 T^*M),$$

where $C = \underset{x \in supp(\phi)}{max} |R_-^{\frac{1}{2}}(x)|$. It suffices then to prove that the operator $\phi H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^*M)$. We recall that we have a compact embedding between the Sobolev space $W^{1,2}(\Lambda^1 T^*K)$ and the space $L^2(\Lambda^1 T^*M)$ for all compact subsets K of M (see [46] p.24, 27, 34). Since ϕ has compact support and $Im(H^{-\frac{1}{2}}) = \mathcal{D}(\overrightarrow{\mathfrak{h}}) \subseteq W^{1,2}(\Lambda^1 T^*M)$, we deduce that the operator $\phi H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^*M)$.
We conclude that $\Lambda = (R_-^{\frac{1}{2}} H^{-\frac{1}{2}})^* (R_-^{\frac{1}{2}} H^{-\frac{1}{2}})$ is compact on $L^2(\Lambda^1 T^*M)$. $\square$

We are now able to end the proof of **Theorem 3.5.1**.

Proof of Theorem 3.5.1. First we notice that $\overrightarrow{\Delta}$ being positive on $L^2(\Lambda^1 T^*M)$, we have for all $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}})$

$$(R_-\omega, \omega) \leq (H\omega, \omega).$$

Then for all $\omega \in L^2(\Lambda^1 T^*M)$

$$(\Lambda\omega, \omega) = (H^{-\frac{1}{2}} R_- H^{-\frac{1}{2}}\omega, \omega) \leq \|\omega\|_2^2.$$

Hence

$$\|\Lambda\|_{2-2} \leq 1. \tag{3.42}$$

According to the self-adjointness and the positivity of Λ, we have

$$\|\Lambda\|_{2-2} = \max\{\lambda; \lambda \,\text{eigenvalue of}\, \Lambda\}. \tag{3.43}$$

Furthermore, **Lemma 3.5.10** and the Fredholm alternative imply

$$1 \,\text{is an eigenvalue of}\ \Lambda \iff Ker_{L^2}(I - \Lambda) \neq \{0\}, \tag{3.44}$$

whereas **Lemma 3.5.9** ensures that

$$Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) = \{0\} \iff Ker_{L^2}(I - \Lambda) = \{0\}. \tag{3.45}$$

Therefore we deduce from (3.42), (3.43), (3.44) and (3.45) that

$$Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) = \{0\} \iff \|\Lambda\|_{2-2} < 1.$$

Since Λ is self-adjoint on $L^2(\Lambda^1 T^*M)$, note that

$$R_- \text{ is } \epsilon\text{-sub-critical} \iff \exists\, 0 \leq \epsilon < 1, \|\Lambda\|_{2-2} \leq \epsilon.$$

The result follows.

$\square$

The following results aim at removing the assumption $Ker_{\mathcal{D}(\overrightarrow{\mathfrak{h}})}(\overrightarrow{\Delta}) = \{0\}$. However we need to strengthen the assumption on $\|R_-^{\frac{1}{2}}\|_{vol}$. We start with a proposition.

Proposition 3.5.11. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then there exists a non-negative constant C depending on the constants appearing in (D) and (G) such that for any $\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}})$*

$$(R_-\omega, \omega) \leq C\|R_-^{\frac{1}{2}}\|_{vol}^2 \overrightarrow{\mathfrak{h}}(\omega, \omega) = C\|R_-^{\frac{1}{2}}\|_{vol}^2 (H\omega, \omega).$$

Proof. We have

$$(R_-\omega,\omega)=\|R_-^{\frac{1}{2}}\omega\|_2^2=\|R_-^{\frac{1}{2}}H^{-\frac{1}{2}}H^{\frac{1}{2}}\omega\|_2^2\leq\|R_-^{\frac{1}{2}}H^{-\frac{1}{2}}\|_{2-2}^2\|H^{\frac{1}{2}}\omega\|_2^2.$$

Using **Lemma 3.5.5**, we obtain the desired result. □

An immediate consequence of **Proposition 3.5.11** is the following.

Proposition 3.5.12. *Suppose that the assumptions* (D) *and* (G) *are satisfied and that* $\|R_-^{\frac{1}{2}}\|_{vol}$ *is small enough. Then* R_- *satisfies* (*S-C*).

In the particular case of polynomial volume growth, we then ask $\|R_-\|_{\frac{N}{2}-\eta}$ and $\|R_-\|_{\frac{N}{2}+\eta}$ to be small enough for some $\eta>0$ to have R_- satisfying (S-C). Note that if M satisfies a global Sobolev inequality, it is easy to prove that R_- satisfies (S-C) if $\|R_-\|_{\frac{N}{2}}$ is small enough (without any assumption on the volume growth).
Note also that we recover $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})=\{0\}$ with the assumptions of **Proposition 3.5.12** but we did not need to assume it to prove subcriticality.

Chapter 4

The Hodge-de Rham Laplacian and L^p-boundedness of Riesz transforms on non-compact manifolds

Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property as well as a Gaussian upper bound for the corresponding heat kernel. We study the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ on both Hardy spaces H^p and Lebesgue spaces L^p under two different conditions on the negative part of the Ricci curvature R_-. First we prove that if R_- is α-subcritical for some $\alpha \in [0, 1)$, then the Riesz transform $d^* \overrightarrow{\Delta}^{-\frac{1}{2}}$ on differential 1-forms is bounded from the associated Hardy space $H^p_{\overrightarrow{\Delta}}(\Lambda^1 T^* M)$ to $L^p(M)$ for all $p \in [1, 2]$. As a consequence, $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1, p_0)$ where $p_0 > 2$ depends on α and the constant appearing in the doubling property. Second, we prove that if

$$\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{p_1}}} \right\|_{p_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} < \infty,$$

for some $p_1 > 2$ and $p_2 > 3$, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $1 < p < p_2$. Furthermore, we study the boundedness of the Riesz transform of Schrödinger operators $A = \Delta + V$ on L^p for $p > 2$ under conditions on R_- and the potential V. We prove both positive and negative results on the boundedness of $dA^{-\frac{1}{2}}$ on L^p.

4.1 Introduction

Let (M, g) be a complete non-compact Riemannian manifold and let ρ be the geodesic distance and μ be the Riemannian measure associated with the metric g. Assume that M satisfies the volume doubling property, that is, there exists a constant $C > 0$ such that

$$v(x, 2r) \leq Cv(x, r) \text{ for all } x \in M,\ r \geq 0,$$

where $v(x, r)$ denotes the volume of the ball $B(x, r)$ of center x and radius r. This property is equivalent to the following one. There exist constants $C > 0$ and $D > 0$ such that

$$v(x, \lambda r) \leq C\lambda^D v(x, r) \ \text{ for all } x \in M,\ r \geq 0,\ \lambda \geq 1. \tag{4.1}$$

Note that the constant D is not unique and (4.1) holds with any $D' > D$. In many cases it is suitable to take D as small as possible in (4.1).

Let Δ be the non-negative Laplace-Beltrami operator on M and $p_t(x, y)$ the corresponding heat kernel, i.e., the integral kernel of the semigroup $e^{-t\Delta}$. We assume that $p_t(x, y)$ satisfies a Gaussian upper bound

$$p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} e^{-c\frac{\rho^2(x,y)}{t}} \text{ for all } t > 0,\ x, y \in M, \tag{4.2}$$

where $c, C > 0$ are constants. The validity of (4.2) has been intensively studied in the literature, see [25, 33, 35, 13] and the references therein.

We consider the Riesz transform $d\Delta^{-\frac{1}{2}}$. Integration by parts shows that $d\Delta^{-\frac{1}{2}}$ is bounded from $L^2(M)$ to $L^2(\Lambda^1 T^* M)$, where $\Lambda^1 T^* M$ denotes the space of differential 1-forms. We address the problem whether the Riesz transform $d\Delta^{-\frac{1}{2}}$ could be extended to a bounded operator from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \neq 2$.

Under the assumptions (4.1) and (4.2), it was proved by Coulhon and Duong [20] that $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1, 2]$. They also gave a counter-example which shows that (4.1) and (4.2) are not sufficient in the case $p > 2$. So additional assumptions are needed. Many works have been devoted to this problem.
Under Li-Yau estimates, or equivalently under the doubling condition and a L^2 Poincaré inequality, Auscher and Coulhon [4] proved that there exists $\epsilon > 0$ such that $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $2 \leq p < 2 + \epsilon$. In the same setting, Auscher, Coulhon, Duong and Hofmann [5] found an equivalence between the boundedness of the Riesz transform on L^p for $p > 2$ and the gradient estimate $\|de^{-t\Delta}\|_{p-p} \leq C/\sqrt{t}$ for the corresponding semigroup on L^p.
Bakry [8] proved that if the manifold has a non-negative Ricci curvature, then $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1, \infty)$.

Another idea to treat the case $p > 2$ is by duality. By the commutation formula $\overrightarrow{\Delta} d = d\Delta$, we are then interested in the Riesz transform $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ where $\overrightarrow{\Delta} = dd^* + d^*d$ is the Hodge-de Rham Laplacian on differential 1-forms. The boundedness of $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ on L^q for some $q \in (1,2)$ implies the boundedness of $d\Delta^{-\frac{1}{2}}$ on L^p where $\frac{1}{p} + \frac{1}{q} = 1$. This strategy was used in Coulhon and Duong [21] by looking at the heat kernel on differential forms. They also made an interesting connection between boundedness of the Riesz transform and Littlewood-Paley-Stein inequalities.

Let us recall the Bochner formula

$$\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_-$$

where R_+ and R_- are respectively the positive and negative part of the Ricci curvature and ∇ denotes the Levi-Civita connection on M. We use this formula to look at $\overrightarrow{\Delta}$ as a Schrödinger operator on 1-forms and then to bring known techniques for the Riesz transforms of Schrödinger operators on functions and try to adapt them to this setting. We note however that the boundedness of the Riesz transform of a Schrödinger operator is a delicate task even in the Euclidean setting. See Assaad and Ouhabaz [2] and also the last sections of the present paper.

In the first part of this paper, we assume that the negative part R_- is α-subcritical for some $\alpha \in [0,1)$, that is

$$0 \le (R_-\omega, \omega) \le \alpha(H\omega, \omega) := \alpha((\nabla^*\nabla + R_+)\omega, \omega) \text{ for all } \omega \in C_c^\infty(\Lambda^1 T^*M), \tag{4.3}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Lambda^1 T^*M)$. We prove that $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is bounded from $H^1_{\overrightarrow{\Delta}}(\Lambda^1 T^*M)$ to $L^1(M)$ where $H^1_{\overrightarrow{\Delta}}(\Lambda^1 T^*M)$ is the Hardy space associated with the operator $\overrightarrow{\Delta}$, see Section 4.2 for details and definitions. The boundedness of $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ from $H^1_{\overrightarrow{\Delta}}(\Lambda^1 T^*M)$ to $L^1(M)$ can also be obtained by combining the results in [6] and [7] by Auscher, McIntosh, Russ and Morris. Here we give a somewhat direct proof by using the Davies-Gaffney estimates. By interpolation, it follows that $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^*M)$ to $L^p(M)$ for all $p \in (1,2]$ if $D \le 2$ and all $p \in (p_0', 2]$ if $D > 2$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$. The latter result was proved recently by Magniez [43]. As a corollary, the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1,\infty)$ if $D \le 2$ and all $p \in (1, p_0)$ if $D > 2$.
The above L^p-boundedness result is not sharp in general. Note that if α is close to 1 or when D is large, then p_0 is close to 2. In [28, Theorem 14], Devyver proved that if M satisfies the classical Sobolev inequality together with the additional assumption that balls of large radius have a polynomial volume growth, that is $cr^D \le v(x,r) \le Cr^D$ for $r \ge 1$, and $R_- \in L^{\frac{D}{2}-\varepsilon} \cap L^\infty$ for some $\varepsilon > 0$, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from

$L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, D)$. Our aim is to prove a similar result without assuming the Sobolev inequality. We suppose that the negative part of the Ricci curvature R_- satisfies

$$\|R_-\|_{vol} := \int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_1}}} \right\|_{p_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} < \infty, \tag{4.4}$$

for some $p_1 > 2$ and $p_2 > 3$. We prove that the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $1 < p < p_2$. In particular, if $v(x,r) \geq Cr^{D_\infty}$ for all $r \geq 1$ and some $D_\infty > 3$, then the condition $\int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} < \infty$ holds if $R_- \in L^{\frac{D_\infty}{2}-\eta}$ for some $\eta > 0$. Similarly, if $v(x,r) \geq Cr^{D_0}$ for $r \in (0,1)$, then $\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_1}}} \right\|_{p_1} \frac{dt}{\sqrt{t}} < \infty$ holds if $R_- \in L^{\frac{D_0}{2}+\eta'}$ for some $\eta' > 0$. In these cases, the condition $\|R_-\|_{vol} < \infty$ is satisfied if $R_- \in L^{\frac{D_0}{2}+\eta'} \cap L^{\frac{D_\infty}{2}+\eta}$ for some $\eta, \eta' > 0$. Therefore, we obtain that the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $1 < p < D_\infty - \eta$. This latter result recovers and extends a result of Devyver [28] who assumes $R_- \in L^{\frac{D}{2}-\eta} \cap L^\infty$ together with the Sobolev inequality. Note that if the Sobolev inequality is satisfied, $R_- \in L^{\frac{D}{2}-\eta} \cap L^{\frac{D}{2}+\eta}$, R_- is strongly sub-critical and $v(x,r) \geq Cr^D$ for all $r > 0$, then [28, Theorem 11] implies a Gaussian upper bound for the heat kernel of $\overrightarrow{\Delta}$ and this implies the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ on L^p for all $p \in (1, \infty)$. Let us also mention recent results by Carron [16] who proved in particular that if the negative part of the Ricci curvature has a quadratic decay and the volume satisfies a reverse doubling condition with a "dimension" $\nu > 2$, then $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for $p \in (1, \nu)$.

In the last two sections of the paper we consider the Riesz transform of Schrödinger operators. Let $A = \Delta + V$ with signed potential $V = V^+ - V^-$. Similarly to our first result, we assume that $V^+ \in L^1_{loc}$ and V^- satisfies α-subcritical condition for some $\alpha \in [0,1)$:

$$\int_M V^- u^2 d\mu \leq \alpha \Big[\int_M |\nabla u|^2 d\mu + \int_M V^+ u^2 d\mu \Big] \text{ for all } u \in W^{1,2}(M). \tag{4.5}$$

Under this assumption on the potential V, we prove that the associated Riesz transform $dA^{-\frac{1}{2}}$ is bounded from $H^1_A(M)$ to $L^1(\Lambda^1 T^*M)$. By interpolation we obtain that $dA^{-\frac{1}{2}}$ is bounded on $L^p(M)$ for all $p \in (1,2]$ if $D \leq 2$ and all $p \in (p_0', 2]$ if $D > 2$, where again $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$. The latter result was proved by Assaad and Ouhabaz [2] by a different approach. For $p > 2$, we assume in addition that the negative part of the Ricci curvature R_- satisfies (4.4) and also V satisfies (4.4) that is

$$\int_0^1 \left\| \frac{|V|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_1}}} \right\|_{p_1} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|V|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} < \infty. \tag{4.6}$$

Then we prove that $dA^{-\frac{1}{2}}$ is bounded on L^p for all $p_0' < p < \frac{p_0 r}{p_0+r}$ where $r = \inf(p_1, p_2)$. In the particular case where the volume $v(x,r)$ has polynomial growth and $V^- = 0$, our result implies that $dA^{-\frac{1}{2}}$ is bounded on L^p for all $1 < p < D$ provided $V \in L^{\frac{D}{2}-\eta} \cap L^{\frac{D}{2}+\eta}$ for some $\eta > 0$. In the last section we prove that the interval $(1, D)$ cannot be improved in general. More precisely, we assume that the manifold M satisfies the Poincaré inequality, the doubling condition (4.1) with a constant D and that there exists a positive bounded function ϕ such that $A\phi = 0$. We then prove that if the Riesz transform $dA^{-\frac{1}{2}}$ is bounded on L^p for some $p > D$, then $V = 0$. A similar result was proved by Guillarmou and Hassell [36] on complete non-compact and asymptotically conic manifolds and assuming V smooth and sufficiently vanishing at infinity. In particular this is satisfied on the Euclidean space $\mathbb{R}^n$ with a smooth and compactly supported potential V.

Throughout, the symbols "c" and "C" will denote (possibly different) constants that are independent of the essential variables.

4.2 Hardy spaces associated with self-adjoint operators

In this preparatory section, we recall the definition of Hardy spaces H_L^p associated with a given operator L on the manifold M. The operator L is either acting on functions or differential 1-forms. The Hardy spaces H_L^p associated with operators on metric measured spaces have been studied by several authors, see for example [3, 6, 7, 29, 38, 39, 40].

Let (X, ρ, μ) be a metric measured space satisfying the doubling condition (4.1). We suppose that X has a smooth structure which allows to define a smooth vector bundle TX over X. In the next section, X will be a Riemannian manifold M, $TX = \Lambda^1 T^* M$ and $L = \overrightarrow{\Delta}$. For $f(x) \in T_x X$ we use the usual notation $|f(x)|_x^2 = (f(x), f(x))_x$ but we will also write $|\cdot|$ instead of $|\cdot|_x$. We recall the definitions of the finite speed propagation property and the Davies-Gaffney estimates for a semigroup e^{-tL}.
For $r > 0$, we set

$$\mathcal{D}_r := \{(x,\, y) \in X \times X : \rho(x,\, y) \le r\}.$$

Given an operator T on $L^2(TX)$, we write

$$\operatorname{supp} K_T \subseteq \mathcal{D}_r \tag{4.7}$$

if $\langle Tf_1, f_2 \rangle = 0$ for all $f_k \in C(TX)$ with $\operatorname{supp} f_k \subseteq B(x_k, r_k)$ for $k = 1, 2$ and $r_1 + r_2 + r < \rho(x_1, x_2)$. If T is an integral operator with kernel K_T, then (4.7) has the usual meaning $K_T(x,y) = 0$ for a.e. $(x,y) \in X \times X$ with $\rho(x,y) > r$.

Definition 4.2.1. Given a non-negative self-adjoint operator L on $L^2(TX)$, one says that the operator L satisfies the *finite speed propagation property* if

$$\text{(FS)} \qquad \operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t \quad \text{for all } t \ge 0\,.$$

Definition 4.2.2. One says that the semigroup $\{e^{-tL}\}_{t>0}$ generated by (minus) L satisfies the *Davies-Gaffney estimates* if there exist constants C, $c > 0$ such that for all open subsets U_1, $U_2 \subset X$ and all $t > 0$,

$$\text{(DG)} \qquad |\langle e^{-tL} f_1, f_2\rangle| \leq C \exp\left(-\frac{\operatorname{dist}(U_1, U_2)^2}{c\,t}\right) \|f_1\|_{L^2(TX)} \|f_2\|_{L^2(TX)},$$

for every $f_i \in L^2(TX)$ with supp $f_i \subset U_i$, $i = 1, 2$, where $\operatorname{dist}(U_1, U_2) := \inf_{x\in U_1, y\in U_2} \rho(x, y)$.

The following result is taken from [48, Theorem 2].

Proposition 4.2.3. *Let L be a non-negative self-adjoint operator acting on $L^2(TX)$. Then the finite speed propagation property* (FS) *and Davies-Gaffney estimates* (DG) *are equivalent.*

Next we recall the definition of Hardy spaces associated with self-adjoint operators. Assume that the operator L satisfies the Davies-Gaffney estimates (DG). Following [6, 7, 29, 38] one can define the L^2 adapted Hardy space by

$$H^2(TX) := \overline{R(L)}, \tag{4.8}$$

that is, the closure of the range of L in $L^2(TX)$. Then $L^2(TX)$ is the orthogonal sum of $H^2(TX)$ and the null space $N(L)$. Consider the following quadratic functional associated to L :

$$S_K f(x) := \Big(\int_0^\infty\!\!\int_{\rho(x,y)<t} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{d\mu(y)}{v(x,t)} \frac{dt}{t}\Big)^{\frac{1}{2}}, \tag{4.9}$$

where $x \in X$, $f \in L^2(TX)$ and K is a natural number. For each $K \geq 1$ and $0 < p < \infty$, we now define

$$D_{K,p} := \Big\{f \in H^2(TX) :\ S_K f \in L^p(X)\Big\}.$$

Definition 4.2.4. Let L be a non-negative self-adjoint operator on $L^2(TX)$ satisfying the Davies-Gaffney estimates (DG).

(i) For each $p \in (0, 2]$, the Hardy space $H^p_L(TX)$ associated with L is the completion of the space $D_{1,p}$ with respect to the norm

$$\|f\|_{H^p_L(TX)} := \|S_1 f\|_{L^p(X)}.$$

(ii) For each $p \in (2, \infty)$, the Hardy space $H^p_L(TX)$ associated with L is the completion of the space $D_{K_0,p}$ in the norm

$$\|f\|_{H^p_L(TX)} := \|S_{K_0} f\|_{L^p(X)}, \quad \text{where } K_0 = \Big[\,\frac{D}{4}\,\Big] + 1.$$

It can be verified that the dual of $H_L^p(TX)$ is $H_L^{p'}(TX)$, with $\frac{1}{p}+\frac{1}{p'}=1$ (see Proposition 9.4 of [38]). We also have complex interpolation and Marcinkiewicz-type interpolation results between $H_L^p(TX)$ for $1 \le p \le 2$ (see Proposition 9.5 and Theorem 9.7 [38]). Although the above results in [38] are stated and proved on X and not on TX, the whole machinery developed there works in the context of Hardy spaces over TX.
Note that if we only assume Davies-Gaffney estimates on the heat kernel of L, for $1 < p < \infty$, $p \neq 2$, $H_L^p(TX)$ may or may not coincide with the space $L^p(TX)$. For the relation of $H_L^p(TX)$ and $L^p(TX)$ for $1 < p \le 2$, we have the following proposition.

Proposition 4.2.5. *Let L be an injective, non-negative self-adjoint operator on $L^2(TX)$ satisfying the finite propagation speed property* (FS) *and general* $(p_0, 2)$*-Davies-Gaffney estimates*

$$(\mathrm{DG_{p_0}}) \qquad \|\chi_{B(x,t)} e^{-t^2 L} \chi_{B(y,t)}\|_{p_0 \to 2} \le C v(x,t)^{\frac{1}{2}-\frac{1}{p}} \exp\Big(- c\rho(x,y)^2/t^2 \Big)$$

for some p_0 with $1 \le p_0 \le 2$. Then for each p with $p_0 < p \le 2$, the Hardy space $H_L^p(TX)$ and the Lebesgue space $L^p(TX)$ coincide and their norms are equivalent.

Proof. This proof is the same as for Hardy space $H_L^p(X)$ (see [40, Proposition 9.1(v)] and [3]). For more details, see also [54, Theorem 4.19]. □

We denote by $\mathcal{D}(L)$ the domain of the operator L. The following definition of atoms of Hardy spaces associated with operators was introduced in [38].

Definition 4.2.6. Let M be a positive integer. A function $a \in L^2(TX)$ is called a $(1, 2, M)$-atom associated with L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B such that

(i) $a = L^M b$,

(ii) supp $L^k b \subset B, \ k = 0, 1, \dots, M$,

(iii) $||(r_B^2 L)^k b||_{L^2} \le r_B^{2M} v(B)^{-\frac{1}{2}}, \ k = 0, 1, \dots, M$, where $v(B)$ is the volume of the ball B.

We can define the atomic Hardy space $H^1_{L,at,M}(TX)$ as follows. First, we say that $f = \sum \lambda_j a_j$ is an atomic $(1, 2, M)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^1$, each a_j is a $(1, 2, M)$-atom and the sum converges in $L^2(TX)$. Then set

$$\mathbb{H}^1_{L,at,M}(TX) := \{f : \ f \text{ has an atomic } (1,2,M)\text{-representation}\},$$

with the norm given by

$$\|f\|_{\mathbb{H}^1_{L,at,M}(TX)} := \inf\{\sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is an atomic } (1,2,M)\text{-representation}\}.$$

The space $H^1_{L,at,M}(TX)$ is then defined as the completion of $\mathbb{H}^1_{L,at,M}(TX)$ with respect to this norm. According to [38, Theorem 2.5], if $M > D/4$ and L satisfies Davies-Gaffney estimate (DG), then the Hardy space $H^1_L(TX)$ coincides with the atomic Hardy space $H^1_{L,at,M}(TX)$ and their norms are equivalent.

4.3 Boundedness of Riesz transforms on Hardy spaces of forms

To prove the boundedness of the Riesz transform on Hardy spaces associated with self-adjoint operators on forms, we need the following lemma.

Lemma 4.3.1. *Assume that T is a non-negative sublinear operator and bounded from $L^2(TX)$ to $L^2(X)$. Also assume that for every $(1,2,M)$-atom a, we have*

$$\|Ta\|_{L^1(X)} \leq C$$

with constant C independent of a. Then T is bounded from $H^1_L(TX)$ to $L^1(X)$.

Proof. For the proof, we refer the reader to [38, Lemma 4.3 and Proposition 4.13]. □

We now state a criterion that allows to derive estimates on Hardy spaces $H^p_L(TX)$. It is already stated in [30, Theorem 3.1] for spectral multipliers $T = m(L)$ on $H^p_L(X)$. We show that the arguments there are valid for a general linear operator T which is bounded on L^2. We do not require the commutation of T with the semigroup e^{-tL}.

Let $U_j(B) = 2^{j+1}B \setminus 2^jB = U_j$ when $j \geq 2$ and $U_1(B) = 4B$.

Lemma 4.3.2. *Let L be a non-negative self-adjoint operator acting on $L^2(TX)$ and satisfying the Davies-Gaffney estimates* (DG)*. Let T be a linear operator which is bounded from $L^2(TX)$ to $L^2(X)$. Assume that there exist constants $M \geq 1$, $s > D/2$ and $C > 0$ such that for every $j = 1, 2, \dots,$*

$$\|T(I - e^{-r^2L})^M f\|_{L^2(U_j(B))} \leq C2^{-js}\|f\|_{L^2(B)} \tag{4.10}$$

for every ball B with radius r and for all $f \in L^2(TX)$ with supp $f \subset B$. Then the operator T extends to a bounded operator from $H^1_L(TX)$ to $L^1(X)$.

Proof. Let $a = L^M b$ be a $(1,2,M)$-atom. By **Lemma 4.3.1**, it is enough to prove that

$$\|Ta\|_{L^1(X)} \leq C \tag{4.11}$$

with constant C independent of the atom a.
Denote $B := B(x,r)$ the ball containing the support of the atom a. We have

$$\|Ta\|_{L^1(X)} \leq \sum_{j=1}^{\infty} \|Ta\|_{L^1(U_j(B))}.$$

Note that by Hölder's inequality and L^2-boundedness of the operator T

$$\|Ta\|_{L^1(4B)} \leq v(4B)^{\frac{1}{2}}\|Ta\|_{L^2(X)} \leq Cv(B)^{\frac{1}{2}}\|a\|_{L^2(TX)}.$$

By (iii) of **Definition 4.2.6**, $\|a\|_{L^2} \leq v(B)^{-\frac{1}{2}}$ and thus

$$\|Ta\|_{L^1(4B)} \leq Cv(B)^{\frac{1}{2}}v(B)^{-\frac{1}{2}} \leq C. \tag{4.12}$$

Then we only need to prove that there exist some constants $\varepsilon > 0$ and $C > 0$ independent of the atom a such that

$$\|Ta\|_{L^1(U_j(B))} \leq C2^{-j\varepsilon} \tag{4.13}$$

for $j = 2, 3, \ldots$
Following (8.7) and (8.8) in [39] or (3.5) in [30], we write

$$\begin{aligned} I &= 2(r^{-2}\int_r^{\sqrt{2}r} tdt) \cdot I \\ &= 2r^{-2}\int_r^{\sqrt{2}r} t(I - e^{-t^2L})^M dt + \sum_{\alpha=1}^{M} C_{j,M} r^{-2}\int_r^{\sqrt{2}r} te^{-jt^2L}dt, \end{aligned}$$

where $C_{\alpha,M}$ are some constants depending only on α and M only. Using the fact that $\partial_t e^{-\alpha t^2 L} = -2\alpha t L e^{-\alpha t^2 L}$ and applying the procedure M times, we have for every function f on TX,

$$\begin{aligned} f &= 2^M\Big(r^{-2}\int_r^{\sqrt{2}r} t(I - e^{-t^2L})^M dt\Big)^M f \\ &+ \sum_{\beta=1}^{M} r^{-2\beta}(I - e^{-r^2L})^\beta \Big(r^{-2}\int_r^{\sqrt{2}r} t(I - e^{-t^2L})^M dt\Big)^{M-\beta} \sum_{\alpha=1}^{(2M-1)\beta} C_{\beta,\alpha,M} e^{-\alpha r^2 L} L^{-\beta} f \\ &:= \sum_{\beta=0}^{M-1} r^{-2\beta} r^{-2}\int_r^{\sqrt{2}r} tF_{\beta,M,r}(L)(I - e^{-t^2L})^M L^{-\beta} f dt \\ &\quad + r^{-2M}F_{M,M,r}(L)(I - e^{-r^2L})^M L^{-M} f \end{aligned} \tag{4.14}$$

where

$$F_{\beta,M,r}(L) = (I - e^{-r^2L})^\beta \Big(r^{-2}\int_r^{\sqrt{2}r} t(I - e^{-t^2L})^M dt\Big)^{M-\beta-1} \sum_{\alpha=1}^{(2M-1)\beta} C_{\beta,\alpha,M} e^{-\alpha r^2 L}$$

for $0 \leq \beta \leq M - 1$ and

$$F_{M,M,r}(L) = \sum_{\alpha=1}^{(2M-1)M} C_{M,\alpha,M} e^{-\alpha r^2 L}.$$

It follows from the Davies-Gaffney estimates that the operator $F_{\beta,M,r}(L)$, $\beta = 0, 1, \cdots, M$, satisfies L^2 off-diagonal estimates, which means that there exist some constants $c, C > 0$ such that

$$\|F_{\beta,M,r}(L)f\|_{L^2(U_j(B))} \leq Ce^{-c4^{|j-i|}}\|f\|_{L^2(U_i(B))}. \tag{4.15}$$

For the details, see [30, pp. 307-309].
By (i) of **Definition 4.2.6**, $a = L^M b$. Then applying (4.14), we have

$$\begin{aligned} Ta &= \sum_{\beta=0}^{M-1} r^{-2\beta} r^{-2} \int_r^{\sqrt{2}r} tT(I - e^{-t^2L})^M F_{\beta,M,r}(L) L^{M-\beta} b dt \\ &\quad + r^{-2M} T(I - e^{-r^2L})^M F_{M,M,r}(L) b. \end{aligned}$$

Then by Hölder's inequality

$$\begin{aligned} &\|Ta\|_{L^1(U_j(B))} \\ &\leq Cv(U_j(B))^{\frac{1}{2}} \sum_{\beta=0}^{M} r^{-2\beta} \sup_{t\in[r,\sqrt{2}r]} \|T(I - e^{-t^2L})^M F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_j(B))}. \end{aligned} \tag{4.16}$$

For $t \in [r, \sqrt{2}r]$, let $B^t := B(x, t)$. Then

$$\begin{aligned} &\|T(I - e^{-t^2L})^M F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_j(B^t))} \\ &\leq \sum_{i=1}^{\infty} \|T(I - e^{-t^2L})^M \chi_{U_i(B^t)} F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_j(B^t))}. \end{aligned} \tag{4.17}$$

For $|i - j| \leq 4$, by L^2 boundedness of $T(I - e^{-t^2L})^M$ and off-diagonal estimates (4.15),

$$\begin{aligned} &\|T(I - e^{-t^2L})^M \chi_{U_i(B^t)} F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_j(B^t))} \\ &\leq \|F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_i(B^t))} \\ &\leq Ce^{-c4^i} \|L^{M-\beta} b\|_{L^2}. \end{aligned} \tag{4.18}$$

For $i \leq j-4$, we decompose $U_i(B^t)$ as the union of a finite number of balls $B_{\kappa,i} = B(x_{\kappa,i}, t)$, the number is compared with 2^{iD} and $\operatorname{dist}(B_{\kappa,i}, B) \geq C2^i r$. For each $B_{\kappa,i}$, we can write

$$U_j(B^t) \subset \bigcup_{\ell=0}^{2i} U_{j-i+\ell}(B_{\kappa,i}).$$

Thus by condition (4.10) and off-diagonal estimates (4.15),

$$\begin{aligned} &\|T(I - e^{-t^2L})^M \chi_{U_i(B^t)} F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_j(B^t))} \\ &\leq \sum_\kappa \sum_\ell \|T(I - e^{-t^2L})^M \chi_{B_{\kappa,i}} F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(U_{j-i+\ell}(B_{\kappa,i}))} \\ &\leq C \sum_\kappa \sum_\ell 2^{-(j-i+\ell)s} \|F_{\beta,M,r}(L) L^{M-\beta} b\|_{L^2(B_{\kappa,i})} \\ &\leq C \sum_\kappa \sum_\ell 2^{-(j-i+\ell)s} e^{-c4^i} \|L^{M-\beta} b\|_{L^2} \end{aligned}$$

$$\leq \quad C2^{iD}2^{-(j-i)s}e^{-c4^i}\|L^{M-\beta}b\|_{L^2}.$$

Thus

$$\begin{aligned}
&\sum_{i=1}^{j-4}\|T(I-e^{-t^2L})^M\chi_{U_i(B^t)}F_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(U_j(B^t))}\\
&\quad\leq \quad C2^{-js}\|L^{M-\beta}b\|_{L^2}\sum_{i=1}^{j-4}2^{iD+is}e^{-c4^i}\leq C2^{-js}\|L^{M-\beta}b\|_{L^2}. \qquad (4.19)
\end{aligned}$$

For $i\geq j+4$, decompose $U_i(B^t)$ as the union of finite number of balls $B_{\kappa,i}=B(x_{\kappa,i},t)$, the number is compared with 2^{iD} and $\operatorname{dist}(B_{\kappa,i},B)\geq C2^ir$. For any $B_{\kappa,i}$, we can write

$$U_j(B^t)\subset\bigcup_{\ell=-2}^{2j+1}U_{i-j+\ell}(B_{\kappa,i}).$$

Thus by condition (4.10) and off-diagonal estimates (4.15),

$$\begin{aligned}
&\|T(I-e^{-t^2L})^M\chi_{U_i(B^t)}F_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(U_j(B^t))}\\
&\leq \quad \sum_{\kappa}\sum_{\ell}\|T(I-e^{-t^2L})^M\chi_{B_{\kappa,i}}F_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(U_{i-j+\ell}(B_{\kappa,i}))}\\
&\leq \quad C\sum_{\kappa}\sum_{\ell}2^{-(i-j+\ell)s}\|F_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(B_{\kappa,i})}\\
&\leq \quad C\sum_{\kappa}\sum_{\ell}2^{-(i-j+\ell)s}e^{-c4^i}\|L^{M-\beta}b\|_{L^2}\\
&\leq \quad C2^{iD}2^{-(i-j)s}e^{-c4^i}\|L^{M-\beta}b\|_{L^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i=j+4}^{\infty}\|T(I-e^{-t^2L})^M\chi_{U_i(B^t)}F_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(U_j(B^t))}\\
&\quad\leq \quad C2^{js}\|L^{M-\beta}b\|_{L^2}\sum_{i=j+4}^{\infty}2^{iD-is}e^{-c4^i}\leq C2^{-js}\|L^{M-\beta}b\|_{L^2}. \qquad (4.20)
\end{aligned}$$

Combining the estimates (4.17), (4.18), (4.19) and (4.20), it follows that

$$\|T(I-e^{-t^2L})^MF_{\beta,M,r}(L)L^{M-\beta}b\|_{L^2(U_j(B^t))}\leq C2^{-js}\|L^{M-\beta}b\|_{L^2}.$$

Noting that for $t\in[r,\sqrt{2}r]$, we have

$$U_j(B)\subset U_j(B^t)\cup U_{j-1}(B^t).$$

Thus, by (4.16) and by (iii) of **Definition 4.2.6**,

$$\begin{aligned}\|Ta\|_{L^1(U_j(B))} &\le Cv(U_j(B))^{\frac{1}{2}}r^{-2\beta}2^{-js}\|L^{M-\beta}b\|_{L^2}\\ &\le C2^{-js}v(U_j(B))^{\frac{1}{2}}r^{-2\beta}r^{2\beta}v(B)^{-\frac{1}{2}}\\ &\le C2^{-j(s-D/2)},\end{aligned}$$

which proves (4.13). Then combining estimate (4.12), we complete the proof of **Lemma 4.3.2**. □

Now we state the main result of this section.

Theorem 4.3.3. *Let M be a complete non-compact Riemannian manifold satisfying assumptions* (4.1) *with a constant D, Gaussian upper bound* (4.2) *and α-subcritical condition* (4.3)*. Then the associated Riesz transform $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is*

i) bounded from $H^1_{\overrightarrow{\Delta}}(\Lambda^1T^*M)$ to $L^1(M)$,

ii) bounded from $H^p_{\overrightarrow{\Delta}}(\Lambda^1T^*M)$ to $L^p(M)$ for all $p \in [1,2]$,

iii) bounded from $L^p(\Lambda^1T^*M)$ to $L^p(M)$ for all $p \in (1,2]$ if $D \le 2$ and all $p \in (p_0',2]$ if $D > 2$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$.

Proof. We apply **Lemma 4.3.2**. Let $X = M$, $TX = \Lambda^1T^*M$ and $L = \overrightarrow{\Delta}$.
The estimate (4.10) was proved in the proof of Theorem 1.1 in [43] (the proof of estimate (34) in Page 23). This gives assertion i).
Assertion ii) follows from i) by interpolation and the fact that $d^*\overrightarrow{\Delta}^{-\frac{1}{2}}$ is bounded from $L^2(\Lambda^1T^*M)$ to $L^2(M)$.
Finally, the Davies-Gaffney estimate (DG_p) was proved in [43], Theorem 4.1. We then apply **Proposition 4.2.5** to obtain iii). □

By duality and the commutation formula

$$\overrightarrow{\Delta}d = d\Delta \tag{4.21}$$

we obtain the following corollary of **Theorem 4.3.3**.

Corollary 4.3.4. *Let M be a complete non-compact Riemannian manifold satisfying assumptions* (4.1) *with a constant D, Gaussian upper bound* (4.2) *and α-subcritical condition* (4.3)*. Then the associated Riesz transform $d\Delta^{-\frac{1}{2}}$ is*

i) bounded from $L^p(M)$ to $H^p_{\overrightarrow{\Delta}}(\Lambda^1T^*M)$ for all $p \in [2,\infty)$,

ii) bounded from $L^p(M)$ to $L^p(\Lambda^1T^*M)$ for all $p \in [2,\infty)$ if $D \le 2$ and all $p \in [2,p_0)$ if $D > 2$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$.

As mentioned in the introduction, assertion ii) was already proved in [43].

4.4 The Riesz transform for $p > 2$

Our aim in this section is to investigate the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ on L^p for other values of $p > 2$ which are not covered by the previous corollary. In order to do this we make an integrability assumption on the Ricci curvature.

Our main result in this section is the following theorem.

Theorem 4.4.1. *Assume that the Riemannian manifold M satisfies the doubling condition* (4.1) *and the Gaussian upper bound* (4.2)*. Assume that the negative part R_- of the Ricci curvature R satisfies* (4.4) *for some p_1 and p_2 such that $p_2 > 3$. Then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $1 < p < p_2$.*

Remark 4.4.2. Suppose that $v(x,r) \geq v(r)$ for all $r > 0$. Then (4.4) is satisfied for p_1 and p_2 such that

$$\int_0^1 v(t)^{-1/p_1} dt < \infty \text{ with } R_- \in L^{\frac{p_1}{2}} \text{ and } \int_1^\infty v(t)^{-1/p_2} dt < \infty \text{ with } R_- \in L^{\frac{p_2}{2}}.$$

Before we start the proof of the theorem we state the following result on $L^p - L^q$ estimates for perturbations of $\overrightarrow{\Delta}$ by a non-negative potential. The manifold M satisfies the same assumptions as in the previous theorem.

Proposition 4.4.3. *Let $\mathcal{R} = \mathcal{R}^+ - \mathcal{R}^-$ be a field of symmetric endomorphisms acting on $\Lambda^1 T^*M$. Let $\alpha \in [0,1)$. We suppose that $\mathcal{R}^-$ is α-subcritical for the operator $\nabla^*\nabla + \mathcal{R}^+$, that is for all $\omega \in \mathcal{C}_c^\infty(\Lambda^1 T^*M)$*

$$(\mathcal{R}^-\omega, \omega) \leq \alpha((\nabla^*\nabla + \mathcal{R}^+)\omega, \omega).$$

Then for every open subsets E and F of M
i) $\|\chi_E d^* e^{-t(\nabla^*\nabla + \mathcal{R}^+ - \mathcal{R}^-)}\chi_F\|_{2\to 2} \leq \frac{C}{\sqrt{t}} e^{-c\frac{dist(E,F)^2}{t}}$
ii) $\|e^{-t(\nabla^*\nabla + \mathcal{R}^+ - \mathcal{R}^-)}\chi_{B(x,r)}\|_{p\to q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r})^\beta$,
where C, c and β are positive constants. The assertion ii) holds for all $p \leq q \in (1,\infty)$ if $D \leq 2$ and all $p \in (p_0', p_0), q \in [p, p_0)$ if $D > 2$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$.

If $\mathcal{R}$ is the Ricci curvature, this theorem was proved in [43], Theorem 4.1 and Corollary 4.5. In the general case, the proof is the same as in [43].

*Proof of **Theorem 4.4.1**.* We shall proceed in three main steps. Let us point out that we may always assume that $D > p_2$ since we can take D as large as we wish in (4.1).

Step I. For a fixed point $x_0 \in M$, we prove that there exist a positive number r_0 sufficiently large and a positive function $W \in C_c^\infty(M)$ with $\operatorname{supp} W \subset B(x_0, r_0)$ such that the following $L^p - L^q$ estimates hold for all $p \in (p_0', p_0)$, $q \in [p, p_0)$ and $t > 1$

$$\|e^{-t(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{p\to q} \le C_{x_0,r_0} t^{-p_2(\frac{1}{p}-\frac{1}{q})/2}.$$

Here p_0 is as in the previous proposition in which the number α will be chosen later. Given $\varepsilon > 0$, by the Dominated Convergence Theorem we can find a large enough r_0 such that

$$\int_0^1 \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_1}}} \right\|_{L^{p_1}(B(x_0,\frac{r_0}{2})^c)} \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{|R_-|^{\frac{1}{2}}}{v(\cdot,\ \sqrt{t})^{\frac{1}{p_2}}} \right\|_{L^{p_2}(B(x_0,\frac{r_0}{2})^c)} \frac{dt}{\sqrt{t}} < \varepsilon. \tag{4.22}$$

We construct a function $0 \le \varphi \in C_c^\infty(M)$ such that $\varphi = 1$ in the ball $B(x_0, r_0/2)$, $\varphi \le 1$ in $B(x_0, r_0)\backslash B(x_0, r_0/2)$ and $\varphi = 0$ outside the ball $B(x_0, r_0)$. Let $V = \varphi|R_-|$. Then V is a compactly supported function and smooth except when $|R_-| = 0$. We choose another function W such that $W \ge V$, $W \in C_c^\infty(M)$ and $\operatorname{supp} W \subset B(x_0, r_0)$. We write

$$\overrightarrow{\Delta} + W = \nabla^*\nabla + R_+ + (W - R_-)^+ - (W - R_-)^-.$$

Note that $(W - R_-)^- = (W - V + V - R_-)^- \le (V - R_-)^-$ and $|(V - R_-)^-| = 0$ in the ball $B(x_0, r_0/2)$ and $|(V - R_-)^-| \le |R_-|$ outside the ball $B(x_0, r_0/2)$. Hence it follows from (4.22) that $\|(W - R_-)^-\|_{vol} < \varepsilon$. Using Proposition 5.8 in [43] with $(W - R_-)^-$ and $\nabla^*\nabla + R_+ + (W - R_-)^+$ instead of R_- and $\nabla^*\nabla + R_+$, we obtain

$$((W - R_-)^-\omega, \omega) \le C\|(W - R_-)^-\|_{vol}^2((\nabla^*\nabla + R_+ + (W - R_-)^+)\omega, \omega).$$

Then we choose ε small enough in (4.22) so that $C\|(W - R_-)^-\|_{vol} < C\varepsilon < 1$. It follows that $(W - R_-)^-$ is ε-subcritical with respect to $\nabla^*\nabla + R_+ + (W - R_-)^+$, i.e.

$$((W - R_-)^-\omega, \omega) \le \varepsilon((\nabla^*\nabla + R_+ + (W - R_-)^+)\omega, \omega) \text{ for all } \omega \in C_c^\infty(\Lambda^1 T^*M). \tag{4.23}$$

By **Proposition 4.4.3**, we obtain $L^p - L^q$ off-diagonal estimates for $e^{-t(\overrightarrow{\Delta}+W)}$ for all $p \in (p_0', p_0), q \in [p, p_0)$ with $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$. Hence for $p \in (p_0', p_0), q \in [p, p_0)$ and $\sqrt{t} \ge r_0$

$$\|e^{-t(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{p\to q} \le \|e^{-t(\overrightarrow{\Delta}+W)}\chi_{B(x_0,\sqrt{t})}\|_{p\to q} \le Cv(x_0, \sqrt{t})^{-(\frac{1}{p}-\frac{1}{q})}, \tag{4.24}$$

and for $1 < \sqrt{t} \le r_0$,

$$\|e^{-t(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{p\to q} \le \frac{C}{v(x_0, r_0)^{\frac{1}{p}-\frac{1}{q}}}(\frac{r_0}{\sqrt{t}})^\beta \le C_{x_0,r_0} t^{-p_2(\frac{1}{p}-\frac{1}{q})/2}. \tag{4.25}$$

Note that $|V| \leq |R_-|$ and so $\|V\|_{vol} \leq \|R_-\|_{vol} \leq C$, which gives

$$\int_1^\infty \left\| \frac{V^{\frac{1}{2}}}{v(\cdot,\sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} \leq C.$$

Since $\operatorname{supp} V \subset B(x_0, r_0)$, we have for $x \in B(x_0, r_0)$ and $\sqrt{t} > r_0$

$$v(x,\sqrt{t}) = \frac{v(x,\sqrt{t})}{v(x_0,\sqrt{t})} v(x_0,\sqrt{t}) \leq C \left(1 + \frac{\rho(x,x_0)}{\sqrt{t}}\right)^D v(x_0,\sqrt{t}) \leq C' v(x_0,\sqrt{t}).$$

Thus

$$\|V^{\frac{1}{2}}\|_{p_2} \int_{r_0^2}^\infty \frac{1}{v(x_0,\sqrt{t})^{\frac{1}{p_2}}} \frac{dt}{\sqrt{t}} \leq C' \int_{r_0^2}^\infty \left\| \frac{V^{\frac{1}{2}}}{v(\cdot,\sqrt{t})^{\frac{1}{p_2}}} \right\|_{p_2} \frac{dt}{\sqrt{t}} \leq C.$$

Note that if $V = 0$, then $R_- = 0$ in the ball $B(x_0, r_0/2)$ and so $\|R_-\|_{vol} < \varepsilon$ by (4.22). As a consequence, R_- satisfies the ε-subcritical condition (4.3). By Corollary 1.2 in [43] or our **Corollary 4.3.4**, $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for $p \in (1, p_0)$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$. With our choice of ε, we have $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for $p \in (1, p_2)$[1]. In the sequel, we assume that $V \neq 0$. Hence

$$\int_{r_0}^\infty \frac{1}{v(x_0,s)^{\frac{1}{p_2}}} ds \leq \int_{r_0^2}^\infty \frac{1}{v(x_0,\sqrt{t})^{\frac{1}{p_2}}} \frac{dt}{\sqrt{t}} \leq C \|V^{\frac{1}{2}}\|_{p_2}^{-1} \leq C'.$$

Let $f(s) = v(x_0,s)^{-\frac{1}{p_2}}$. Note that f is a positive continuous decreasing function. So by the first mean value theorem, there exists $\xi \in [r_0, t]$ such that

$$\int_{r_0}^t f(s)ds = f(\xi)(t - r_0) \geq f(t)t - f(r_0)r_0.$$

We deduce that for $t > r_0$

$$0 < tf(t) \leq r_0 f(r_0) + \int_{r_0}^t f(s)ds \leq r_0 f(r_0) + \int_{r_0}^\infty f(s)ds \leq C.$$

That is for $t > r_0$

$$f(t) \leq C/t$$

and hence

$$v(x_0, t) \geq C t^{p_2}. \tag{4.26}$$

[1] if $R_- = 0$, then the Ricci curvature is non-negative and it is well known that the Riesz transform is bounded on L^p for all $p \in (1, \infty)$

Therefore by (4.24), (4.25) and (4.26), we have the following estimate for $p \in (p_0', p_0), q \in [p, p_0)$ and all $t > 1$

$$\|e^{-t(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{p\to q} \le C_{x_0,r_0} t^{-p_2(\frac{1}{p}-\frac{1}{q})/2}. \tag{4.27}$$

Step II. We prove that the operator $d(\Delta+W)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in [2, p_2)$. In order to do this we take the difference with $(\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d$, that is,

$$d(\Delta+W)^{-\frac{1}{2}} = d(\Delta+W)^{-\frac{1}{2}} - (\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d + (\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d.$$

It follows from (4.23) and [43, Theorem 1.1] or our **Theorem 4.3.3** that $d^*(\overrightarrow{\Delta}+W)^{-\frac{1}{2}}$ is bounded on $L^p(\Lambda^1T^*M)$ to $L^p(M)$ for all $p \in (p_0', 2]$ with again $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$. By duality, $(\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d$ is bounded from $L^p(M)$ to $L^p(\Lambda^1T^*M)$ for all $p \in [2, p_0)$. Choosing again ε small enough such that $p_0 \ge p_2$, it follows that $(\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d$ is bounded from $L^p(M)$ to $L^p(\Lambda^1T^*M)$ for all $p \in [2, p_2)$.
It remains to prove the boundedness of $d(\Delta+W)^{-\frac{1}{2}} - (\overrightarrow{\Delta}+W)^{-\frac{1}{2}}d$. For this part we follow the strategy in [28, Section 5.2] and [15, Section 3.2]. In these two papers, the authors assume a global Sobolev inequality on the manifold together with a polynomial lower bound on the volume. We adapt their ideas to our setting.
Let $\Box_t = -\frac{\partial^2}{\partial t^2} + (\overrightarrow{\Delta}+W)$. We have for $u \in C_c^\infty(M)$

$$\Box_t\Big(de^{-t\sqrt{\Delta+W}}u - e^{-t\sqrt{\overrightarrow{\Delta}+W}}du\Big) = \Box_t de^{-t\sqrt{\Delta+W}}u = -(e^{-t\sqrt{\Delta+W}}u)dW.$$

The operator $L_1 := -\frac{\partial^2}{\partial t^2}$ with domain

$$D(L_1) = W^{2,2}((0,\infty), L^2(\Lambda^1T^*M)) \cap W_0^{1,2}((0,\infty), L^2(\Lambda^1T^*M))$$

is self-adjoint on $L^2((0,\infty), L^2(\Lambda^1T^*M))$. We define L_2 as an "extension" of $\overrightarrow{\Delta}+W$ to $L^2((0,\infty), L^2(\Lambda^1T^*M))$ in the following usual way

$$(L_2w)(t,x) := (\overrightarrow{\Delta}+W)(w(t,\cdot))(x)$$

with domain

$$D(L_2) := \{w \in L^2((0,\infty), L^2(\Lambda^1T^*M)) : w(t,x) \in D(\overrightarrow{\Delta}+W) \text{ for a.e. t}\}.$$

The operators L_1 and L_2 are self-adjoint and commute. Therefore, $e^{-sL_1}e^{-sL_2} = e^{-sL_2}e^{-sL_1}$ is a strongly continuous semigroup whose generator $\mathcal{C}$ is the closure of L_1+L_2 on the domain $D(L_1) \cap D(L_2)$ (see for example [31], p. 64).

Let $\phi = de^{-t\sqrt{\Delta+W}}u - e^{-t\sqrt{\vec{\Delta}+W}}du$. Assume for a moment that $\phi \in D(L_1) \cap D(L_2)$ and that $\mathcal{C}$ is injective. Then

$$\mathcal{C}\phi = \Box_t \phi = -(e^{-t\sqrt{\Delta+W}}u)dW,$$

and we have

$$\begin{aligned}
de^{-t\sqrt{\Delta+W}}u - e^{-t\sqrt{\vec{\Delta}+W}}du &= \phi = -\mathcal{C}^{-1}((e^{-t\sqrt{\Delta+W}}u)dW) \\
&= -\int_0^\infty e^{-s\mathcal{C}}((e^{-t\sqrt{\Delta+W}}u)dW)ds \\
&= -\int_0^\infty e^{-sL_1}e^{-sL_2}((e^{-t\sqrt{\Delta+W}}u)dW)ds \\
&= \int_0^\infty \int_0^\infty K_s(t,\sigma)e^{-s(\vec{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)d\sigma ds,
\end{aligned}$$

where

$$K_s(t,\sigma) = \frac{e^{-\frac{(\sigma+t)^2}{4s}} - e^{-\frac{(\sigma-t)^2}{4s}}}{\sqrt{4\pi s}}$$

is the heat kernel on the half-line $\mathbb{R}_+$ for the Dirichlet boundary condition at 0.
Next we write

$$\begin{aligned}
&d(\Delta+W)^{-\frac{1}{2}}u - (\vec{\Delta}+W)^{-\frac{1}{2}}du \\
&\quad = \int_0^\infty (de^{-t\sqrt{\Delta+W}}u - e^{-t\sqrt{\vec{\Delta}+W}}du)dt \\
&\quad = \int_0^\infty \int_0^\infty \int_0^\infty K_s(t,\sigma)e^{-s(\vec{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)d\sigma ds dt =: G(u).
\end{aligned}$$

Since $W \geq 0$ and $e^{-t\Delta}$ satisfies the Gaussian upper bound (4.2), it follows that $e^{-t(\Delta+W)}$ also satisfies the same bound (this follows from the domination property $|e^{-t(\Delta+W)}f| \leq e^{-t\Delta}|f|$). Therefore,

$$\|e^{-t(\Delta+W)}\chi_{B(x_0,r_0)}\|_{p\to 2} \leq Cv(x_0,\sqrt{t})^{-(\frac{1}{p}-\frac{1}{2})}(1+\frac{r_0}{\sqrt{t}})^D.$$

It follows from the subordination formula $e^{-t\sqrt{A}} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}e^{-\frac{t^2A}{4u}}du$ that

$$\|e^{-t\sqrt{\Delta+W}}\chi_{B(x_0,r_0)}\|_{p\to 2} \leq Cv(x_0,t)^{-(\frac{1}{p}-\frac{1}{2})}(1+\frac{r_0}{t})^D.$$

By (4.26) and (4.1), we have for all $p \in [1,2]$ and $t > 1$,

$$\|e^{-t\sqrt{\Delta+W}}\chi_{B(x_0,r_0)}\|_{p\to 2} \leq C_{x_0,r_0}t^{-p_2(\frac{1}{p}-\frac{1}{2})}, \tag{4.28}$$

and similarly for all $p \geq 2$ and $t \geq 1$

$$\|\chi_{B(x_0,r_0)}e^{-t\sqrt{\Delta+W}}\|_{p\to\infty} \leq C_{x_0,r_0}t^{-\frac{p_2}{p}}. \tag{4.29}$$

From these estimates we want to obtain that $\|G(u)\|_p \leq C\|u\|_p$ for all $p \in [2, p_2)$. Since these estimates are valid for $t > 1$ we have to treat first the case of small t and s in the definition of G.
Let $g_{s,t}(u) := e^{-s(\overrightarrow{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)$. Then

$$\begin{aligned}
G(u) &= \int_0^\infty \int_0^\infty \int_0^\infty K_s(t,\sigma) g_{s,t}(u) d\sigma ds dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\frac{(\sigma+t)^2}{4s}} - e^{-\frac{(\sigma-t)^2}{4s}}}{\sqrt{4\pi s}} g_{s,t}(u) d\sigma ds dt \\
&= \int_0^\infty \int_0^\infty \frac{g_{s,t}(u)}{\sqrt{4\pi s}} \Big[\int_0^\infty e^{-\frac{(\sigma+t)^2}{4s}} - e^{-\frac{(\sigma-t)^2}{4s}} d\sigma \Big] ds dt \\
&= -\int_0^\infty \int_0^\infty \frac{g_{s,t}(u)}{\sqrt{4\pi s}} \Big[\int_{-t}^t e^{-\frac{\sigma^2}{4s}} d\sigma \Big] ds dt \\
&= -\int_0^\infty \int_0^\infty \frac{2 g_{s,t}(u)}{\sqrt{\pi}} \Big[\int_0^{\frac{t}{2\sqrt{s}}} e^{-\gamma^2} d\gamma \Big] ds dt \\
&= -\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-\gamma^2} \Big[\int_0^{\frac{t^2}{4\gamma^2}} g_{s,t}(u) ds \Big] d\gamma dt.
\end{aligned}$$

Thus

$$\|G(u)\|_{L^p} \leq \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-\gamma^2} \Big[\int_0^{\frac{t^2}{4\gamma^2}} \|g_{s,t}(u)\|_{L^p} ds \Big] d\gamma dt. \tag{4.30}$$

For all $s \in [0,1]$ and $t \in [0,1]$, by the fact that the semigroup $e^{-s(\overrightarrow{\Delta}+W)}$ and $e^{-t\sqrt{\Delta+W}}$ are uniformly bounded on L^p for all $s > 0$ and $t > 0$,

$$\|g_{s,t}(u)\|_{L^p} = \|e^{-s(\overrightarrow{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)\|_{L^p} \leq C\|dW\|_{L^\infty}\|u\|_{L^p}.$$

For all $s \in [0,1]$ and $t > 1$, by the fact that the semigroup $e^{-s(\overrightarrow{\Delta}+W)}$ is uniformly bounded on L^p for all $s > 0$ and estimate (4.29),

$$\begin{aligned}
\|g_{s,t}(u)\|_{L^p} &= \|e^{-s(\overrightarrow{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)\|_{L^p} \\
&\leq C\|(e^{-t\sqrt{\Delta+W}}u)dW\|_{L^p} \\
&\leq C\|dW\|_{L^p}\|\chi_{B(x_0,r_0)} e^{-t\sqrt{\Delta+W}}u\|_{L^\infty} \\
&\leq CC_{x_0,r_0}\|dW\|_{L^p} t^{-\frac{p_2}{p}} \|u\|_{L^p}.
\end{aligned}$$

For all $s > 1$ and $t \in [0,1]$, by the fact that the semigroup $e^{-t\sqrt{\Delta+W}}$ is uniformly bounded on L^p for all $t > 0$ and estimate (4.27),

$$\|g_{s,t}(u)\|_{L^p} = \|e^{-s(\overrightarrow{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)\|_{L^p}$$

$$
\begin{aligned}
&\le C\|e^{-s(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{(p_0'+\varepsilon)\to p}\|(e^{-t\sqrt{\Delta+W}}u)dW\|_{L^{p_0'+\varepsilon}} \\
&\le CC_{x_0,r_0}s^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|dW\|_{L^{1/(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})}}\|e^{-t\sqrt{\Delta+W}}u\|_{L^p} \\
&\le CC_{x_0,r_0}\|dW\|_{L^{1/(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})}}s^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|u\|_{L^p}.
\end{aligned}
$$

Similarly, for all $s > 1$ and $t > 1$,

$$
\begin{aligned}
\|g_{s,t}(u)\|_{L^p} &= \|e^{-s(\overrightarrow{\Delta}+W)}((e^{-t\sqrt{\Delta+W}}u)dW)\|_{L^p} \\
&\le C\|e^{-s(\overrightarrow{\Delta}+W)}\chi_{B(x_0,r_0)}\|_{(p_0'+\varepsilon)\to p}\|(e^{-t\sqrt{\Delta+W}}u)dW\|_{L^{p_0'+\varepsilon}} \\
&\le CC_{x_0,r_0}s^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|dW\|_{L^{1/(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})}}\|\chi_{B(x_0,r_0)}e^{-t\sqrt{\Delta+W}}u\|_{L^p} \\
&\le CC_{x_0,r_0}\|dW\|_{L^{1/(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})}}s^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|\chi_{B(x_0,r_0)}\|_{L^p}\|\chi_{B(x_0,r_0)}e^{-t\sqrt{\Delta+W}}u\|_{L^\infty} \\
&\le CC_{x_0,r_0}\|dW\|_{L^{1/(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})}}s^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|\chi_{B(x_0,r_0)}\|_{L^p}t^{-\frac{p_2}{p}}\|u\|_{L^p}.
\end{aligned}
$$

Combining the above four estimates, we get

$$
\|g_{s,t}(u)\|_{L^p} \le C(1+t)^{-\frac{p_2}{p}}(1+s)^{-p_2(\frac{1}{p_0'+\varepsilon}-\frac{1}{p})/2}\|u\|_{L^p}.
$$

Putting this estimate into estimate (4.30) and noting that $p > 2$, $p_2 > 3$ and $p < p_2$, we have

$$
\|G(u)\|_{L^p} \le C\|u\|_{L^p}.
$$

Hence, $d(\Delta+W)^{-\frac{1}{2}}$ is bounded on L^p for $p \in [2, p_2)$.

Recall that we have used in the previous proof that $\phi \in D(L_1)\cap D(L_2)$ and $\mathcal{C}$ is injective. Now we prove these two properties.
If $\mathcal{C}\psi = 0$, then $\psi = e^{-sL_1}e^{-sL_2}\psi = e^{-sL_2}e^{-sL_1}\psi$ and the self-adjointness of L_1 and L_2 imply that $\psi \in D(L_1)\cap D(L_2)$. Hence

$$
\langle \frac{\partial^2}{\partial t^2}\psi, \psi\rangle_{L^2(\mathbb{R}_+\times TM)} = 0,
$$

which implies $\partial_t\psi = 0$ and thus $\psi(x,t) = \psi(x)$. In addition, $\psi \in W_0^{1,2}$ implies $\psi(x,0) = 0$ and hence $\psi = 0$. This shows that $\mathcal{C}$ is injective.
Now we prove that $\phi \in D(L_1)\cap D(L_2)$. For fixed t, because u and W belong to C_c^∞, it is easy to see that $\phi \in L^2$ and $(\overrightarrow{\Delta}+W)\phi \in L^2$. Thus $\phi \in D(L_2)$. Note that

$$
\lim_{t\to 0}\phi = \lim_{t\to 0} de^{-t\sqrt{\Delta+W}}u - \lim_{t\to 0} e^{-t\sqrt{\overrightarrow{\Delta}+W}}du = du - du = 0.
$$

($\lim_{t\to 0} de^{-t\sqrt{\Delta+W}}u = du$ comes from that fact $d(\Delta+W)^{-1/2}$ is bounded on L^2 and $(\Delta+W)^{1/2}e^{-t\sqrt{\Delta+W}}u$ converges to $(\Delta+W)^{1/2}u$). It remains to check that $\phi \in W^{2,2}((0,\infty), L^2(\Lambda^1 T^*M))$. We write $\phi = \phi_1 - \phi_2$ where $\phi_1 := de^{-t\sqrt{\Delta+W}}u$ and $\phi_2 := e^{-t\sqrt{\vec{\Delta}+W}}du$. Then

$$\begin{aligned}
\int_0^\infty \|\phi_1\|_{L^2}^2 dt &= \int_0^\infty \|de^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&= \int_0^\infty \|d(\Delta+W)^{-1/2}(\Delta+W)^{1/2}e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&\leq \int_0^\infty \|(\Delta+W)^{1/2}e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&\leq \int_0^1 \|(\Delta+W)^{1/2}u\|_{L^2}^2 dt + \int_1^\infty \|t(\Delta+W)^{1/2}e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 \frac{dt}{t^2} \\
&\leq \|(\Delta+W)^{1/2}u\|_{L^2}^2 + \int_1^\infty \|u\|_{L^2}^2 \frac{dt}{t^2} \\
&\leq C.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_0^\infty \|\partial_t\phi_1\|_{L^2}^2 dt &= \int_0^\infty \|d\sqrt{\Delta+W}e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&= \int_0^\infty \|d(\Delta+W)^{-1/2}(\Delta+W)e^{-t\sqrt{\Delta+W}}u\|_{L^2(X)}^2 dt \\
&\leq \int_0^\infty \|(\Delta+W)e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&\leq \int_0^1 \|(\Delta+W)u\|_{L^2}^2 dt + \int_1^\infty \|t^2(\Delta+W)e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 \frac{dt}{t^4} \\
&\leq \|(\Delta+W)u\|_{L^2}^2 + \int_1^\infty \|u\|_{L^2}^2 \frac{dt}{t^4} \\
&\leq C,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \|\partial_t^2\phi_1\|_{L^2}^2 dt &= \int_0^\infty \|d(\Delta+W)e^{-t\sqrt{\Delta+W}}u\|_{L^2}^2 dt \\
&\leq \|(\Delta+W)^{3/2}u\|_{L^2}^2 + \int_1^\infty \|u\|_{L^2}^2 \frac{dt}{t^6} \\
&\leq C.
\end{aligned}$$

By the same calculations, we can prove that $\phi_2, \partial_t\phi_2, \partial_t^2\phi_2 \in L^2((0,\infty), L^2(\Lambda^1 T^*M))$. This shows that $\phi \in D(L_1) \cap D(L_2)$.

Step III. We prove that $d\Delta^{-\frac{1}{2}}$ is bounded on L^p. We write

$$d\Delta^{-\frac{1}{2}} = (d\Delta^{-\frac{1}{2}} - d(\Delta+W)^{-\frac{1}{2}}) + d(\Delta+W)^{-\frac{1}{2}}. \tag{4.31}$$

We have proved in the previous step that $d(\Delta+W)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in [2, p_2)$. Now we prove the boundedness of $d\Delta^{-\frac{1}{2}} - d(\Delta+W)^{-\frac{1}{2}}$. Following the ideas in [2, Section 3.6] with $A_0 = \Delta + W$ and $A = \Delta$, we write

$$\begin{aligned}
&d\Delta^{-\frac{1}{2}} - d(\Delta+W)^{-\frac{1}{2}} \\
&\quad = c\int_0^\infty t^{\frac{1}{2}} d(I+tA_0)^{-1} W (I+tA)^{-1} dt \\
&\quad = c\int_0^\infty dA_0^{-\frac{1}{2}} (tA_0)^{\frac{1}{2}} (I+tA_0)^{-\frac{1}{2}} (I+tA_0)^{-\frac{1}{2}} W^{\frac{1}{2}} W^{\frac{1}{2}} (I+tA)^{-1} dt.
\end{aligned}$$

The operator $dA_0^{-\frac{1}{2}} = d(\Delta+W)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in [2, p_2)$. Next, $(tA_0)^{\frac{1}{2}}(I + tA_0)^{-\frac{1}{2}}$ is uniformly bounded (in $t > 0$) on L^p by the holomorphic functional calculus and the fact that A_0 has a Gaussian bound. For the last two terms in the previous integral, it suffices to prove that

$$\int_0^\infty \|W^{\frac{1}{2}} e^{-sL}\|_{p\to p} \frac{ds}{\sqrt{s}} \leq C$$

for all $p \in [2, p_2)$, where L is A_0 or A. Noting that heat kernel of A_0 or A satisfies Gaussian upper bound, so by volume condition (4.26) and doubling condition, we have for all $p \in [2, p_2)$ and $t > 1$,

$$\|\chi_{B(x_0,r_0)} e^{-tL}\|_{p\to\infty} \leq C_{x_0,r_0} t^{-\frac{p_2}{2p}}.$$

Since $W \in C_c^\infty(M)$

$$\int_0^1 \|W^{\frac{1}{2}} e^{-sL}\|_{p\to p} \frac{ds}{\sqrt{s}} \leq C\|W\|_\infty^{\frac{1}{2}} \int_0^1 \|e^{-sL}\|_{p\to p} \frac{ds}{\sqrt{s}} \leq C\|W\|_\infty^{\frac{1}{2}} \int_0^1 \frac{ds}{\sqrt{s}} \leq C,$$

and using $\operatorname{supp} W \subset B(x_0, r_0)$, we deduce that for $p < p_2$

$$\begin{aligned}
\int_1^\infty \|W^{\frac{1}{2}} e^{-sL}\|_{p\to p} \frac{ds}{\sqrt{s}} &\leq C\|W^{\frac{1}{2}}\|_p \int_1^\infty \|\chi_{B(x_0,r_0)} e^{-sL}\|_{p\to\infty} \frac{ds}{\sqrt{s}} \\
&\leq C\|W^{\frac{1}{2}}\|_p \int_1^\infty C_{x_0,r_0} s^{-\frac{p_2}{2p}} \frac{ds}{\sqrt{s}} \leq C.
\end{aligned}$$

For more details about this last step, we refer to [2, Section 3.6]. □

4.5 Riesz transforms of Schrödinger operators

In this section, we give some results on the boundedness of Riesz transforms $dA^{-\frac{1}{2}}$ of Schrödinger operators $A = \Delta + V$ with signed potential $V = V^+ - V^-$.

We start with the following result.

Theorem 4.5.1. *Let M be a complete non-compact Riemannian manifold satisfying assumptions* (4.2) *and* (4.1) *with some constant D. Let A be the Schrödinger operator with signed potential V such that $V^+ \in L^1_{loc}$ and V^- satisfies α-subcritical condition* (4.5)*. Then the associated Riesz transform $dA^{-\frac{1}{2}}$ is*

i): bounded from $H^1_A(M)$ to $L^1(\Lambda^1 T^* M)$,

ii): bounded from $H^p_A(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in [1, 2]$,

iii): bounded on $L^p(M)$ for all $p \in (1, 2]$ if $D \leq 2$ and all $p \in (p_0', 2]$ if $D > 2$ where $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\alpha})}$.

Proof. Under these assumptions on V it is proved in [2] (see the proof of estimate (43) in page 1127) that

$$\left\|dA^{-\frac{1}{2}}(I - e^{-r^2 A})^M f\right\|_{L^2(U_j(B))} \leq C 2^{-js} \|f\|_{L^2(B)} \tag{4.32}$$

for every ball B with radius r and for all $f \in L^2(M)$ with supp $f \subset B$. Here $M \geq 1$, $s > D/2$ and $C > 0$ are constants. Now by **Lemma 4.3.2** we conclude that assertion i) holds.

Assertion ii) follows by interpolation between $H^p_A(M)$.

Assertion iii) follows from ii) by identifying $H^p_A(M)$ and L^p (cf. **Proposition 4.2.5**) since the estimate (DG_p) was proved in Theorem 3.4 in [2]. □

Note that assertion iii) of the previous theorem was already proved in [2].

For $p > 2$, we give a consequence of **Theorem 4.4.1** and [2, Theorem 3.9].

Theorem 4.5.2. *Assume that the Riemannian manifold M satisfies the doubling condition* (4.1) *and the heat kernel of the Laplacian satisfies the Gaussian upper bound* (4.2)*. Assume also that the negative part of the Ricci curvature R_- satisfies* (4.4) *for some $p_2 > 3$. Let A be the Schrödinger operator with signed potential V which satisfies* (4.6) *and* (4.5) *for some $\alpha \in [0, 1)$. Then $dA^{-\frac{1}{2}}$ is bounded on L^p for $p_0' < p < \frac{p_0 r}{p_0 + r}$ where $r = \inf(p_1, p_2)$.*

This result is a combination of **Theorem 4.4.1** and [2, Theorem 3.9]. Indeed it was proved in [2, Theorem 3.9] that $\Delta^{\frac{1}{2}} A^{-\frac{1}{2}}$ is bounded on L^p for $p_0' < p < \frac{p_0 r}{p_0 + r}$ where $r = \inf(p_1, p_2)$ without assumptions on the Ricci curvature. By **Theorem 4.4.1**, $d\Delta^{-\frac{1}{2}}$ is bounded on L^p. The boundedness of $dA^{-\frac{1}{2}}$ follows by composition.

Remark 4.5.3. Suppose that $V^- = 0$ (or equivalently $\alpha = 0$) and $v(x, r) \geq Cr^D$. Then $p_0 = \infty$ and the assumptions in **Theorem 4.5.2** hold if $R_- \in L^{\frac{D}{2}-\eta} \cap L^{\frac{D}{2}+\eta}$ and $V \in L^{\frac{D}{2}-\eta} \cap L^{\frac{D}{2}+\eta}$ for some $\eta > 0$. Thus, the theorem gives that $dA^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1, D)$. Note that one cannot expect a better interval for boundedness of the Riesz transform of Schrödinger operators as we will show in the next section.

4.6 A negative result for the Riesz transform of Schrödinger operators

In this section, we show a negative result for the boundedness of the Riesz transform for Schrödinger operators. We prove even more : on a wide class of Riemannian manifolds, the Riesz transform $d(\Delta+V)^{-\frac{1}{2}}$ is never bounded on L^p for any $p > D$, unless $V = 0$.

A result in this direction was given by Guillarmou and Hassel [36] on complete non-compact and asymptotically conic manifolds of dimension n. They assumed V is non zero, smooth and sufficiently vanishing at infinity. They proved the Riesz transform $d(\Delta + V)^{-\frac{1}{2}}$ is not bounded on L^p for $p > n$ if there exists a L^2 function ψ such that $(\Delta + V)\psi = 0$.

We recall that a Riemannian manifold M satisfies the L^2 Poincaré inequality if there exists a constant $C > 0$ such that for every $f \in W^{1,2}_{loc}(M)$ and every ball $B = B(x, r)$

$$\left(\int_B |f - f_B|^2 d\mu\right)^{\frac{1}{2}} \leq Cr \left(\int_B |df|^2 d\mu\right)^{\frac{1}{2}}, \tag{4.33}$$

where $f_B = \dfrac{1}{\mu(B)} \displaystyle\int_B f d\mu$.

The main result of this section is the following theorem in which we consider for simplicity only non-negative potentials.

Theorem 4.6.1. *Assume that M satisfies the volume doubling condition* (4.1) *and the Poincaré inequality* (4.33)*. Let $0 \leq V \in L^1_{loc}(M)$ and consider the Schrödinger operator $A = \Delta + V$. We suppose that there exists a positive function ϕ bounded on M such that $e^{-tA}\phi = \phi$. If $\|de^{-tA}\|_{p-p} \leq \frac{C}{\sqrt{t}}$ for some $p > \max(D, 2)$, then $V = 0$. In particular, if $dA^{-\frac{1}{2}}$ is bounded on L^p for some $p > \max(D, 2)$, then $V = 0$.*

Remark 4.6.2. The assumption $e^{-tA}\phi = \phi$ for all $t \geq 0$ with ϕ positive bounded was studied by several authors. We give here some references. In the Euclidean setting $M = \mathbb{R}^n$, Simon [49] proved that if the potential V is in $L^{\frac{n}{2}-\eta} \cap L^{\frac{n}{2}+\eta}$ for a certain $\eta > 0$, then the assumption $e^{-tA}\phi = \phi$ for all $t \geq 0$ is equivalent to the fact that V^- satisfies (4.5). With different methods, Grigor'yan [34] and Takeda [53] proved that if M is non-parabolic and satisfies Li-Yau estimates and if the potential V is non-negative and Green-bounded on M, then such a function ϕ exists.

*Proof of **Theorem 4.6.1**.* From **Lemma 4.6.3** below, we have for all $f \in W^{1,p}(M)$ and for almost every $x, x' \in M$ with $\rho(x, x') \leq 1$

$$|f(x) - f(x')| \leq C_{x,p}\|df\|_p. \tag{4.34}$$

Let $f \in C_c^\infty(M)$ and $x, x' \in M$ with $\rho(x,x') \leq 1$ and fix $p > \max(D,2)$ such that $\|de^{-tA}\|_{p-p} \leq \frac{C}{\sqrt{t}}$ for all $t > 0$. From (4.34), we have for all $t > 0$

$$\begin{aligned} |e^{-tA}(v(.,\sqrt{t})^{1-\frac{1}{p}} f)(x) - e^{-tA}(v(.,\sqrt{t})^{1-\frac{1}{p}} f)(x')| &\leq C\|de^{-tA} v(.,\sqrt{t})^{1-\frac{1}{p}} f\|_p \\ &\leq \frac{C}{\sqrt{t}} \|e^{-\frac{t}{2}A} v(.,\sqrt{t})^{1-\frac{1}{p}} f\|_p. \end{aligned} \tag{4.35}$$

Since (4.1) and (4.33) are equivalent to Li-Yau estimates (see [47]), the heat kernel $p_t(x,y)$ of Δ satisfies the Gaussian upper bound (4.2). Since $V \geq 0$, the heat kernel $k_t(x,y)$ of A satisfies also the same Gaussian upper bound. As a consequence, the semigroup e^{-tA} is uniformly bounded on $L^1(M)$ and the operator $e^{-tL}v(.,\sqrt{t})$ is uniformly bounded from $L^1(M)$ to $L^\infty(M)$. An interpolation argument shows that for all $p \in [1,\infty]$ the operator $e^{-tL}v(.,\sqrt{t})^{1-\frac{1}{p}}$ is bounded from $L^1(M)$ to $L^p(M)$ (see e.g. [13, Proposition 2.1.5]).
It follows from this and (4.35) that

$$|e^{-tA}(v(.,\sqrt{t})^{1-\frac{1}{p}} f)(x) - e^{-tA}(v(.,\sqrt{t})^{1-\frac{1}{p}} f)(x')| \leq \frac{C}{\sqrt{t}} \|f\|_1. \tag{4.36}$$

This extends by density to all $f \in L^1(M)$ and gives

$$\left| \int_M (k_t(x,y) - k_t(x',y)) v(y,\sqrt{t})^{1-\frac{1}{p}} f(y) d\mu(y) \right| \leq \frac{C}{\sqrt{t}} \|f\|_1. \tag{4.37}$$

Since the previous inequality is satisfied for all $f \in L^1(M)$, we obtain for a.e. $x, y \in M$ and all $t > 0$

$$|k_t(x,y) - k_t(x',y)| \leq \frac{C}{\sqrt{t}\, v(y,\sqrt{t})^{1-\frac{1}{p}}}. \tag{4.38}$$

Using (4.1) and (4.2) for $k_t(x,y)$ we find

$$\begin{aligned} &|k_t(x,y) - k_t(x',y)| \\ &\leq |k_t(x,y) - k_t(x',y)|^{\frac{1}{2}} \left[k_t(x,y) + k_t(x',y)\right]^{\frac{1}{2}} \\ &\leq \frac{C}{t^{\frac{1}{4}} v(y,\sqrt{t})^{\frac{1}{2}-\frac{1}{2p}}} \left[\frac{C}{v(x,\sqrt{t})^{\frac{1}{2}}} exp(-c\frac{\rho^2(x,y)}{t}) + \frac{C}{v(x',\sqrt{t})^{\frac{1}{2}}} exp(-c\frac{\rho^2(x',y)}{t}) \right] \\ &\leq \frac{C}{t^{\frac{1}{4}} v(y,\sqrt{t})^{1-\frac{1}{2p}}} \left[exp(-c\frac{\rho^2(x,y)}{t}) + exp(-c\frac{\rho^2(x',y)}{t}) \right]. \end{aligned}$$

Therefore,

$$\int_M |k_t(x,y) - k_t(x',y)| d\mu(y) \leq \frac{C}{t^{\frac{1}{4}}} \left[v(x,\sqrt{t})^{\frac{1}{2p}} + v(x',\sqrt{t})^{\frac{1}{2p}} \right]. \tag{4.39}$$

From (4.1) and since $\rho(x,x') \le 1$, we deduce that for $t \ge 1$

$$\int_M |k_t(x,y) - k_t(x',y)| d\mu(y) \le \frac{Cv(x,\sqrt{t})^{\frac{1}{2p}}}{t^{\frac{1}{4}}} \le \frac{Cv(x,1)^{\frac{1}{2p}}}{t^{\frac{1}{4}(1-\frac{D}{p})}}.$$

Furthermore for all $t \ge 1$

$$\begin{aligned}
|\phi(x) - \phi(x')| &= |e^{-tA}\phi(x) - e^{-tA}\phi(x')| \\
&= \left| \int_M (k_t(x,y) - k_t(x',y))\phi(y) d\mu(y) \right| \\
&\le \frac{Cv(x,1)^{\frac{1}{2p}}}{t^{\frac{1}{4}(1-\frac{D}{p})}} \|\phi\|_\infty.
\end{aligned}$$

We let $t \to +\infty$ and since $p > D$ it follows that ϕ is constant on M. From the assumption $e^{-tA}\phi = \phi$ it follows that $A\phi = 0$. The latter equality gives $V\phi = 0$ and finally $V = 0$ since ϕ is positive.
Finally, since the semigroup e^{-tA} is analytic on L^p, if the Riesz transform $dA^{-\frac{1}{2}}$ is bounded on L^p then

$$\|de^{-tA}\|_{p-p} = \|dA^{-\frac{1}{2}}A^{\frac{1}{2}}e^{-tA}\|_{p-p} \le C\|A^{\frac{1}{2}}e^{-tA}\|_{p-p} \le \frac{C'}{\sqrt{t}}.$$

The previous arguments show that $V = 0$. $\square$

To complete the proof of the theorem, it remains to prove the following lemma.

Lemma 4.6.3. *Let $p \ge 2$ and $p > D$. Assume that* (4.1) *and* (4.33) *are satisfied. For all $f \in W^{1,p}(M)$ and for almost every $x, x' \in M$ with $\rho(x,x') \le 1$ there exists a constant $C = C_{x,x',p}$ such that*

$$|f(x) - f(x')| \le C\|df\|_p.$$

Proof. The arguments in this proof are taken from [37, p. 13-14]. We repeat them for the reader's convenience. Write $B_i(x) = B(x, r_i) = B(x, \frac{\rho(x,x')}{2^i})$ for each non-negative integer i and $f_B = \frac{1}{\mu(B)}\int_B f d\mu$. By the Lebesgue differentiation theorem we have for almost every $x \in M$, $f_{B_i(x)} \to f(x)$ as i tends to infinity. Using (4.1), (4.33) and Hölder's inequality we obtain

$$\begin{aligned}
|f(x) - f_{B_0(x)}| &\le \sum_{i=0}^{\infty} |f_{B_{i+1}(x)} - f_{B_i(x)}| \\
&\le \sum_{i=0}^{\infty} \frac{1}{\mu(B_{i+1}(x))} \int_{B_{i+1}(x)} |f - f_{B_i(x)}| d\mu \\
&\le C\sum_{i=0}^{\infty} \frac{1}{\mu(B_i(x))} \int_{B_i(x)} |f - f_{B_i(x)}| d\mu
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{i=0}^{\infty}\left(\frac{1}{\mu(B_i(x))}\int_{B_i(x)}|f-f_{B_i(x)}|^2d\mu\right)^{\frac{1}{2}}\\
&\leq C\sum_{i=0}^{\infty}\frac{\rho(x,x')}{2^i}\left(\frac{1}{\mu(B_i(x))}\int_{B_i(x)}|df|^2d\mu\right)^{\frac{1}{2}}\\
&\leq C\sum_{i=0}^{\infty}\frac{1}{2^i\mu(B_i(x))^{\frac{1}{p}}}\|df\|_p.
\end{aligned}
$$

Using property (4.1) and the fact that $\rho(x,x')\leq 1$, yields

$$
\frac{1}{\mu(B_i(x))^{\frac{1}{p}}}\leq\frac{C2^{i\frac{D}{p}}}{\rho(x,x')^{\frac{D}{p}}v(x,1)^{\frac{1}{p}}}.
$$

Therefore, for $p>D$, we obtain

$$
|f(x)-f_{B_0(x)}|\leq C_{x,x',p}\|df\|_p. \tag{4.40}
$$

Similarly

$$
|f(x')-f_{B_0(x')}|\leq C_{x,x',p}\|df\|_p. \tag{4.41}
$$

Furthermore from the triangle inequality and (4.1) we have

$$
\begin{aligned}
|f_{B_0(x)}-f_{B_0(x')}|&\leq|f_{B_0(x)}-f_{2B_0(x)}|+|f_{B_0(x')}-f_{2B_0(x)}|\\
&\leq\frac{C}{\mu(2B_0(x))}\int_{2B_0(x)}|f-f_{2B_0(x)}|d\mu.
\end{aligned}
$$

We use the same arguments as above and obtain

$$
|f_{B_0(x)}-f_{B_0(x')}|\leq C_{x,x',p}\|df\|_p. \tag{4.42}
$$

The lemma follows combining (4.40), (4.41) and (4.42). $\square$

Chapter 5

L^p estimates of the gradient of the heat semigroup on a Riemannian manifold with Ricci curvature in a Kato class

We consider a complete non-compact Riemannian manifold satisfying the volume doubling property and a Gaussian upper bound for its heat kernel on functions. Let $\vec{\Delta}$ be the Hodge-de Rham Laplacian acting on 1-differential forms. According to the Bochner formula, $\vec{\Delta} = \nabla^*\nabla + R_+ - R_-$ where R_+ and R_- are respectively the positive and negative part of the Ricci curvature and ∇ is the Levi-Civita connection. Under the assumption that R_- is in a Kato class, we prove that for all $p > 2$ and for large t, $\|\nabla e^{-t\Delta}\|_p \leq Ct^{\beta_p - \frac{1}{2}}$ where β_p depends on p and on the constant appearing in the volume doubling property. If in addition one supposes that R_- is ϵ-sub-critical for $\nabla^*\nabla + R_+$, then we prove that the power β_p previously obtained can be improved. The method is based on establishing first a Gaussian upper bound (with an extra power of t) for the heat kernel on 1-differential forms.
We also prove some estimates of the L^p-norm of the semigroup $e^{-t\vec{\Delta}}$ on forms with the same assumptions on R_-.

5.1 Introduction

Let (M, g) be a complete non-compact Riemannian manifold of dimension N, where g denotes a Riemannian metric on M, that is g is a family of smoothly varying positive definite inner products g_x on the tangent space T_xM for each $x \in M$. Let ρ and μ be the Riemannian distance and measure associated with g respectively. We suppose that M

satisfies the volume doubling property, that is there exists constants $C, D > 0$ such that

$$v(x, \lambda r) \leq C\lambda^D v(x, r),\ \forall x \in M, \forall r \geq 0, \forall \lambda \geq 1, \tag{D}$$

where $v(x, r) = \mu(B(x, r))$ denotes the volume of the ball $B(x, r)$ of center x and radius r. This property is equivalent to the following one. For some constant $C > 0$

$$v(x, 2r) \leq Cv(x, r),\ \forall x \in M, \forall r \geq 0.$$

Let Δ be the non-negative Laplace-Beltrami operator. We define Δ with the bilinear form

$$\mathfrak{a}(u, v) = \int_M \nabla u.\nabla v\, d\mu, \forall u, v \in \mathcal{C}_0^\infty(M),$$

$$\mathcal{D}(\mathfrak{a}) = \overline{\mathcal{C}_0^\infty(M)}^{\|\cdot\|_\mathfrak{a}} = W^{1,2}(M),$$

where $\|u\|_\mathfrak{a} = \sqrt{\mathfrak{a}(u, u) + \|u\|_2^2}$. Let $(e^{-t\Delta})_{t\geq 0}$ be the heat semigroup associated with the operator $-\Delta$. We suppose that the heat kernel $p_t(x, y)$ associated with the semigroup $(e^{-t\Delta})_{t\geq 0}$ satisfies a Gaussian upper bound : that is, there exists positive constants c, C such that for all $x, y \in M$ and $t > 0$

$$p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} \exp\left(-c\frac{\rho^2(x, y)}{t}\right). \tag{G}$$

Our motivation in this chapter has its origin in the study of the heat equation

$$\frac{d}{dt}u = -\Delta u,\ \forall t > 0 \text{ and } u(0) = f \in L^p(M).$$

It is well-known that the solution of the above equation is $u(t) = e^{-t\Delta}f$ and that the semigroup $(e^{-t\Delta})_{t\geq 0}$ is a bounded analytic semigroup of contractions on $L^p(M)$ for all $p \in (1, \infty)$, that is

$$\|e^{-t\Delta}\|_p \leq 1,\ \forall t \geq 0, \forall p \in (1, \infty).$$

An interesting question is then to study the $W^{1,p}$-regularity of the solution, which means estimating the L^p-norm of the gradient of $e^{-t\Delta}$. One way to do this is to prove that the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p. Indeed, when this is satisfied for a $p \in (1, \infty)$, the analyticity of the semigroup on $L^p(M)$ and the Cauchy formula imply that

$$\|\nabla e^{-t\Delta}\|_p \leq \frac{C}{\sqrt{t}},\ \forall t \geq 0.$$

The case $p = 2$ is obvious. Indeed, since we have by integration by parts

$$\|d\,f\|_2 = \|\Delta^{\frac{1}{2}} f\|_2, \forall f \in \mathcal{C}_0^\infty(M),$$

the Riesz transform $d\Delta^{-\frac{1}{2}}$ extends to a bounded operator from $L^2(M)$ to $L^2(\Lambda^1 T^*M)$, where $\Lambda^1 T^*M$ denotes the space of 1-forms on M. The boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ on L^p for $p \neq 2$ has been studied by several authors under various assumptions.

Coulhon and Duong [20] proved that under the assumptions (D) and (G), the Riesz transform is bounded on L^p for all $p \in (1, 2]$ and may be not bounded on L^p for $p > 2$. Then additional assumptions are needed to treat the case $p > 2$.

In the sequel we consider $\overrightarrow{\Delta}$ the Hodge-de Rham Laplacian acting on 1-differential forms. According to the Bochner formula, $\overrightarrow{\Delta} = \nabla^*\nabla + R_+ - R_-$ where R_+ and R_- are respectively the positive and negative part of the Ricci curvature and ∇ is the Levi-Civita connection. One can associate to $-\overrightarrow{\Delta}$ a semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ and a heat kernel $\overrightarrow{p_t}(x, y)$.

A few years before the work of Coulhon and Duong [20], Bakry [8] proved, using probabilistic methods, that if $R_- = 0$, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1, \infty)$. From this result, one would wonder under which less restrictive conditions on R_- the Riesz transform could be bounded on L^p for $p > 2$. Several works have been made since to answer this question. See for instance [5], [15], [16], [17], [18], [28], [43].

The purpose of the present chapter is not to obtain the boundedness of the Riesz transform $d\Delta^{-\frac{1}{2}}$ on L^p for $p > 2$. Instead, we study the L^p-norm of $\nabla e^{-t\Delta}$ for $p > 2$ and large t. One of our inspiration is the work of Coulhon and Duong [21]. They proved that, if in addition of (D), (G), one assumes that the heat kernel $\overrightarrow{p_t}(x, y)$ satisfies a Gaussian upper bound, then the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1, \infty)$. One of the key point in their method is to prove pointwise estimates for the gradient of the heat kernel $p_t(x, y)$. More precisely they proved that

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}\, v(x, \sqrt{t})} \exp\left(-c\frac{\rho^2(x, y)}{t}\right), \forall x, y \in M, \forall t > 0.$$

Our method is based on proving analogous pointwise estimates (with a different power of t), which are sufficient to estimate $\|\nabla e^{-t\Delta}\|_p$ for all $p > 2$. This means in particular that there is no need to prove the boundedness of the Riesz transform on L^p to obtain an estimate of$\|\nabla e^{-t\Delta}\|_p$.

In the sequel we will consider R_- in a certain Kato class according to the following definition. This class is a very large class of functions (containing for instance the space of bounded functions).

Definition 5.1.1. $\tilde{K}^N$ denotes the class of functions f satisfying the property : there exists $\xi > 0$ such that

$$\sup_{x \in M} \int_M \left(\int_0^\xi p_s(x,y) ds \right) |f(y)| d\mu(y) < 1.$$

$\tilde{K}^N$ is a subclass of the Kato class K^N of functions satisfying

$$\lim_{\xi \to 0} \sup_{x \in M} \int_M \left(\int_0^\xi p_s(x,y) ds \right) |f(y)| d\mu(y) = 0.$$

The Kato class K^N has played an important role in the study of Schrödinger operators and their associated semigroups. See Simon [50] and the references therein. The class $\tilde{K}^N$ appears in [55]. Voigt studied properties of the semigroups associated to Schrödinger operators with potential in $\tilde{K}^N$ (as $L^p - L^q$ smoothing for instance).

In this chapter, we see $\overrightarrow{\Delta}$ as a Schrödinger operator according to the Bochner formula $\overrightarrow{\Delta} = \nabla^* \nabla + R_+ - R_-$. Since this operator is a vector-valued operator, the techniques for operators acting on functions need modifications. With our main assumption

$$|R_-| \in \tilde{K}^N, \tag{K}$$

we prove that

1. the heat kernel $\overrightarrow{p_t}(x,y)$ satisfies a Gaussian estimate with an extra polynomial term in t
2. this "semi-Gaussian" estimate leads to estimates of $\|\nabla e^{-t\Delta}\|_p$ for all $p > 2$ and $t > 1$
3. this "semi-Gaussian" estimate leads to estimates of $\|e^{-t\overrightarrow{\Delta}}\|_p$ for all $1 \leq p \leq \infty$ and large t.
4. this "semi-Gaussian" estimate implies the boundedness of the local Riesz transforms $d(\Delta + a)^{-\frac{1}{2}}$ on L^p for all $p \in (1, \infty)$, $a > 0$.

Note that condition (K) can also be written in the following probabilistic way

$$\exists \xi > 0 / \sup_{x \in M} \mathbb{E}_x \left[\int_0^\xi |R_-(b_s)| ds \right] < 1,$$

where $(b_s)_s$ denotes the Brownian motion on M.

We now state our main results.

Theorem 5.1.2. *We suppose that the manifold M satisfies the volume doubling condition (D), the Gaussian upper bound for its heat kernel (G) and assumption (K). Then* *(i) there exists $c, C > 0$ and $\alpha \geq 0$ such that for all $t > 0$ and $x, y \in M$*

$$|\overrightarrow{p_t}(x,y)| \leq C\frac{\left(1+\alpha t+\frac{\rho^2(x,y)}{t}\right)^{\frac{D}{2}}}{v(x,\sqrt{t})^{\frac{1}{2}}v(y,\sqrt{t})^{\frac{1}{2}}}\exp\left(-\frac{\rho^2(x,y)}{4t}\right).$$

(ii) for all $t \geq 1$ and $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq C\min\left(1,\frac{t^{\frac{D}{2}}}{v(x,\sqrt{t})}\right)\exp\left(-c\frac{\rho^2(x,y)}{t}\right).$$

(iii) for all $t \geq 1$ and $p \geq 2$

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C_p t^{\left(\frac{1}{2}-\frac{1}{p}\right)D-\frac{1}{2}}.$$

(iv) for all $t > e$ and $p \in [1,\infty]$

$$\|e^{-t\overrightarrow{\Delta}}\|_{p,p} \leq C_p(t\log(t))^{\left|\frac{1}{2}-\frac{1}{p}\right|\frac{D}{2}}.$$

Statement (i) of **Theorem 5.1.2** has the following consequence in term of local Riesz transforms.

Corollary 5.1.3. *We suppose that the manifold M satisfies (D), (G) and (K). Then for all $a > 0$, the local Riesz transform $d(\Delta+a)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1,\infty)$.*

The next result is an improvement of **Theorem 5.1.2** under an additional assumption on R_-. We say that R_- is ϵ-sub-critical if for a certain $\epsilon \in [0,1)$

$$0 \leq (R_-\omega,\omega) \leq \epsilon\,(H\omega,\omega), \forall\omega \in \mathcal{C}_0^\infty(\Lambda^1T^*M),$$

where $H := \nabla^*\nabla + R_+$. For further information on subcriticality, see [26] or [23] and the references therein.

Theorem 5.1.4. *We suppose that the manifold M satisfies the volume doubling condition (D), the Gaussian upper bound for its heat kernel (G) and assumption (K). Suppose in addition that R_- is ϵ-sub-critical. Let $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ and $\tilde{p}_0 = p_0 - \eta$ for any small $\eta > 0$. Then*

(i) there exists $c, C > 0$ such that for all $t > 0$ and $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C(1+t)^{\frac{D}{\tilde{p}_0}}}{v(x,\sqrt{t})}\exp\left(-c\frac{\rho^2(x,y)}{t}\right).$$

(ii) for all $t \geq 1$ and $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq C \min\left(1, \frac{t^{\frac{D}{\tilde{p}_0}}}{v(x,\sqrt{t})}\right) \exp\left(-c\frac{\rho^2(x,y)}{t}\right).$$

(iii) for all $t \geq 1$

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq \frac{C_p}{\sqrt{t}} \text{ if } p \in (1, p_0)$$

and

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C_p t^{\left(\frac{1}{\tilde{p}_0} - \frac{1}{p}\right)D - \frac{1}{2}} \text{ if } p \geq p_0.$$

(iv) for all $t > e$

$$\|e^{-t\overrightarrow{\Delta}}\|_{p,p} \leq C_p \text{ if } p \in (p_0', p_0)$$

and

$$\|e^{-t\overrightarrow{\Delta}}\|_{p,p} \leq C_p (t\log(t))^{\left|\frac{1}{\tilde{p}_0} - \frac{1}{p}\right|\frac{D}{2}} \text{ if } p \in [1, p_0'] \cup [p_0, \infty].$$

A similar result as **Theorem 5.1.4** (*ii*) has been obtained by Coulhon and Zhang [23] but under some restrictive assumptions on the manifold M and R_-. They proved that on a manifold satisfying (D), (G) and $v(x,1) \geq C$ for all $x \in M$, if $R_- \in L^\infty(M)$ and if $|R_-|$ is ϵ-sub-critical for Δ, then

$$|\overrightarrow{p_t}(x,y)| \leq C \min\left(1, \frac{t^{(p-1+\delta)\epsilon}}{v(x,\sqrt{t})}\right) \exp\left(-c\frac{\rho^2(x,y)}{t}\right), \forall x, y \in M, \forall t \geq 1, \forall \delta > 0,$$

provided $R_- \in L^p(M)$ for some $p \geq 2$. Thus, in a certain sense, **Theorem 5.1.4** (*ii*) improves the result of Coulhon and Zhang. Indeed, one can verifiy that $|R_-|$ is ϵ-sub-critical for Δ implies that R_- is ϵ-sub-critical for $\nabla^*\nabla + R_+$. This is a consequence of the properties $\overrightarrow{\Delta} \geq 0$, $R_- \leq |R_-|$ and domination (for more details see [23] p.356). Furthermore, as we observe in the next section, assuming R_- bounded on M is not necessary since $L^\infty(M) \subset \tilde{K}^N$. At last we emphasize that we do not assume $R_- \in L^p(M)$ for any $p \in (1, \infty)$.

Let us add that we learned recently about a work of Coulhon, Devyver and Sikora [19]. It is proved in [19] that on a manifold M satisfying the assumptions (D), (G) and for some $\nu' > 2$

$$v(x, \lambda r) \geq C\lambda^{\nu'} v(x,r), \forall x \in M, r > 0, \lambda \geq 1,$$

if $|R_-|$ satisfies

$$\sup_{x\in M}\int_{M\setminus B(o,A)} g(x,y)|R_-(y)|d\mu(y) < 1,$$

where $o \in M$ is a fixed point, then R_- is ϵ-sub-critical for $\nabla^*\nabla + R_+$ if and only if

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C}{v(x,\sqrt{t})}\exp\left(-c\frac{\rho^2(x,y)}{t}\right), \forall x,y\in M, \forall t>0.$$

This result is sharper than ours but the assumption on R_- in [19] is more restrictive, as it is shown in Section 5.2.

In the present chapter, we recall in Section 5.2 some known results which will be used in the sequel. Section 5.3 is devoted to the proof of **Theorem 5.1.2** and **Corollary 5.1.3**, whereas in Section 5.4 we prove **Theorem 5.1.4**.

5.2 Preliminaries

We first recall well-known results which will be used in the proofs of the main results.

Lemma 5.2.1. *Let $x \in M$. There exists a constant $C > 0$ independent of x such that for all $t > 0$*

$$\int_M \exp\left(-c\frac{\rho^2(x,y)}{t}\right) d\mu(y) \leq Cv(x,\sqrt{t}).$$

Proof. We use a standard decomposition of M into annuli to obtain

$$\begin{aligned}\int_M \exp\left(-c\frac{\rho^2(x,y)}{t}\right) d\mu(y) &\leq \sum_{k=0}^{\infty}\int_{k\sqrt{t}\leq\rho(x,y)\leq(k+1)\sqrt{t}} \exp(-ck^2)d\mu(y)\\ &\leq \sum_{k=0}^{\infty}\exp(-ck^2)v(x,(k+1)\sqrt{t}).\end{aligned}$$

Then the volume doubling property (D) implies the result. □

Lemma 5.2.2. *Let f be a non-negative function in the Kato class $\tilde{K}^N$. Then*

$$\sup_{x\in M}\mathbb{E}_x\left[\exp\left(\int_0^{\xi}|f(b_s)|ds\right)\right] < \infty.$$

Proof. See [50] Lemma B.1.2. □

We now define two more classes of functions in order to prove that the bounded functions are in $\tilde{K}^N$ and that the assumption on R_- appearing in [19] is more restrictive than ours. First we define a local version of $\tilde{K}^N$.

Definition 5.2.3. We say that $f \in \tilde{K}^N_{loc}$ if for all compact set $K \subset M$

$$\lim_{t\to 0} \sup_{x\in M} \int_M \left(\int_0^t p_s(x,y)ds \right) |(\chi_K f)(y)| d\mu(y) = 0. \tag{5.1}$$

Proposition 5.2.4. *We suppose (D) and (G). Then $L^\infty \subset \tilde{K}^N$ and $L^\infty_{loc} \subset \tilde{K}^N_{loc}$. In particular any manifold satisfying (D), (G) and $|R_-| \in L^\infty(M)$ satisfies condition (K).*

Proof. Using (G), $f \in L^\infty$ and **Lemma 5.2.1** gives

$$\begin{aligned}
\int_M \left(\int_0^t p_s(x,y)ds \right) |f(y)| d\mu(y) &\leq \int_0^t \int_M \frac{C}{v(x,\sqrt{s})} \exp\left(-c\frac{\rho^2(x,y)}{s} \right) |f(y)| d\mu(y) ds \\
&\leq \left(\int_0^t \frac{C}{v(x,\sqrt{s})} \int_M \exp\left(-c\frac{\rho^2(x,y)}{s} \right) d\mu(y) ds \right) \|f\|_\infty \\
&\leq Ct \|f\|_\infty,
\end{aligned}$$

where C is a constant independent of x. This proves $L^\infty \subset \tilde{K}^N$. The same argument shows that $L^\infty_{loc} \subset \tilde{K}^N_{loc}$. ☐

Next is the class of functions defined in [19].

Definition 5.2.5. K^∞ denotes the class of functions f satisfying the property : there exists $A > 0$ such that

$$\sup_{x\in M} \int_{M\setminus B(o,A)} g(x,y) |f(y)| d\mu(y) < 1, \tag{5.2}$$

where $o \in M$ is a fixed point.

The following proposition shows that assuming $|R_-| \in K^\infty$ as in [19] is more restrictive than assuming $|R_-| \in \tilde{K}^N$. One reason is that the class K^∞ urges a certain decrease at infinity of its elements whereas $\tilde{K}^N$ does not.

Proposition 5.2.6. *$\tilde{K}^N_{loc} \cap K^\infty \subset \tilde{K}^N$. In particular if $|R_-| \in K^\infty$, then $|R_-| \in \tilde{K}^N$.*

Proof. It suffices to note that for all $A > 0$

$$\begin{aligned}
\int_M \left(\int_0^t p_s(x,y) \right) |f(y)| d\mu(y) \leq& \int_{B(o,A)} \left(\int_0^t p_s(x,y) \right) |f(y)| d\mu(y) \\
&+ \int_{M\setminus B(o,A)} g(x,y) |f(y)| d\mu(y).
\end{aligned}$$

For the second statement, note that $R_- \in L^\infty_{loc}$ and use **Proposition 5.2.4**. ☐

5.3 Proof of Theorem 5.1.2 and Corollary 5.1.3

In this section we prove the four statements of **Theorem 5.1.2** and **Corollary 5.1.3**.

Theorem 5.3.1. *Let M be a Riemannian manifold satisfying the volume doubling property (D), the Gaussian upper bound for its heat kernel (G) and assumption (K). Then there exists $\alpha > 0$ such that the kernel of the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ satisfies the following Gaussian estimate*

$$|\overrightarrow{p_t}(x,y)| \leq \frac{Ce^{\alpha t}}{v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right), \forall t>0, \forall x,y \in M. \tag{5.3}$$

Proof. For simplicity V_- denotes $|R_-|$. The proof is based on the domination $|\overrightarrow{p_t}(x,y)| \leq p_t^{V_-}(x,y)$ where $p_t^{V_-}(x,y)$ is the integral kernel of the semigroup $(e^{-t(\Delta - V_-)})_{t\geq 0}$.

Step 1 We prove that there exists a sufficiently large $\lambda \geq 0$ such that

$$\|(\Delta+\lambda)^{-1}V_-\|_{\infty,\infty} < 1.$$

We have

$$\begin{aligned}
\|(\Delta+\lambda)^{-1}V_-\|_{\infty,\infty} &= \|\int_0^\infty e^{-\lambda s}e^{-s\Delta}V_- ds\|_{\infty,\infty} \\
&\leq \sum_{n=0}^{\infty} \|\int_{n\xi}^{(n+1)\xi} e^{-\lambda s}e^{-s\Delta}V_- ds\|_{\infty,\infty} \\
&\leq \sum_{n=0}^{\infty} \|\int_0^{\xi} e^{-\lambda(t+n\xi)}e^{-(t+n\xi)\Delta}V_- dt\|_{\infty,\infty} \\
&\leq \sum_{n=0}^{\infty} e^{-\lambda n\xi}\|e^{-n\xi\Delta}\|_{\infty,\infty}\|\int_0^{\xi} e^{-\lambda t}e^{-t\Delta}V_- dt\|_{\infty,\infty} \\
&\leq \frac{1}{1-e^{-\lambda\xi}}\|\int_0^{\xi} e^{-t\Delta}V_- dt\|_{\infty,\infty}.
\end{aligned}$$

Since we assume $V_- \in \tilde{K}^N$, we have $\| \int_0^\xi e^{-t\Delta}V_- dt\|_{\infty,\infty} < 1$. It suffices then to choose a λ sufficiently large to have $\|(\Delta+\lambda)^{-1}V_-\|_{\infty,\infty} < 1$. (the argument used above can be found in [55] Proposition 4.7)

Step 2 We prove that there exists $\alpha > 0$ such that the operator $\Delta - V_- + \alpha$ satisfies the following Gagliardo-Nirenberg type inequality

$$\|v(.,\sqrt{t})^{\frac{1}{2}-\frac{1}{q}}u\|_q \leq C(\|u\|_2 + \sqrt{t}\|(\Delta - V_- + \alpha)^{\frac{1}{2}}u\|_2),$$

where $2 < q < +\infty$ is such that $\frac{q-2}{q}D < 2$ and $u \in W^{1,2}(M)$.

Notice that M satisfies the assumptions (D), (G) and that the operator Δ has its semigroup uniformly bounded on $L^1(M)$ and satisfies the $L^2 - L^2$ Davies-Gaffney estimates

$$\|e^{-t\Delta}u\|_{L^2(F)} \leq e^{-\frac{\rho^2(E,F)}{2t}}\|u\|_2,$$

where E, F are two closed subsets of M and $u \in L^2(\Lambda^1 T^*M)$ with support in E. Then applying [13] Theorem 1.2.1, we obtain the following Gagliardo-Nirenberg type inequality

$$\|v(., \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}u\|_q \leq C(\|u\|_2 + \sqrt{t}\|\nabla u\|_2),$$

where $2 < q < +\infty$ is such that $\frac{q-2}{q}D < 2$ and $u \in W^{1,2}(M)$. It suffices then to prove that for some constants $C > 0$ and $\alpha > 0$

$$\|\nabla u\|_2 \leq C\|(\Delta - V_- + \alpha)^{\frac{1}{2}}u\|_2. \tag{5.4}$$

By duality, we deduce from **Step 1** that $\|V_-(\Delta + \lambda)^{-1}\|_{1,1} < 1$. Applying the Stein interpolation theorem to the function $F(z) = V^z(\Delta + \lambda)^{-1}V^{1-z}$, we obtain

$$\|V_-^{\frac{1}{2}}(\Delta + \lambda)^{-\frac{1}{2}}\|_{2,2}^2 = \|V_-^{\frac{1}{2}}(\Delta + \lambda)^{-1}V_-^{\frac{1}{2}}\|_{2,2} < 1.$$

Therefore there exists $\gamma < 1$ such that for all $u \in W^{1,2}(M)$

$$\|V_-^{\frac{1}{2}}u\|_2^2 \leq \gamma\|(\Delta + \lambda)^{\frac{1}{2}}u\|_2^2 = \gamma(\|\nabla u\|_2^2 + \lambda\|u\|_2^2). \tag{5.5}$$

Setting $\alpha = \gamma\lambda$, we easily obtain

$$\|\nabla u\|_2^2 \leq \frac{1}{1-\gamma}(\|\nabla u\|_2^2 - \|V^{\frac{1}{2}}u\|_2^2 + \alpha\|u\|_2^2) = \frac{1}{1-\gamma}\|(\Delta - V_- + \alpha)^{\frac{1}{2}}\|_2^2,$$

which proves (5.4).

Step 3 We prove that there exists $\alpha_0 \geq 0$ such that for all $\alpha \geq \alpha_0$, the semigroup $(e^{-t(\Delta - V_- + \alpha)})_{t\geq 0}$ is uniformly bounded on $L^\infty(M)$.
We follow the ideas of [50] Theorem B.1.1. The Feynman-Kac formula gives for all $t \geq 0$ and $u \in L^\infty(M)$

$$\begin{aligned}\|e^{-t(\Delta - V_-)}u\|_\infty &\leq \sup_{x\in M}\mathbb{E}_x\left[\exp\left(\int_0^t V_-(b_s)ds\right)|u(b_t)|\right] \\ &\leq \sup_{x\in M}\mathbb{E}_x\left[\exp\left(\int_0^t V_-(b_s)ds\right)\right]\|u\|_\infty.\end{aligned}$$

Since $V_- \in \tilde{K}^N$, there exists $\xi > 0$ such that $\sup\limits_{x\in M}\mathbb{E}_x\left[\int_0^\xi V_-(b_s)ds\right] < 1$. By **Lemma 5.2.2** there exists a constant $C_0 > 0$ such that

$$\sup_{0\le t\le \xi} \|e^{-t(\Delta - V_-)}\|_{\infty,\infty} < C_0.$$

Therefore using the semigroup property we deduce that for all $t \ge 0$ and $\delta = t - \xi\lfloor \frac{t}{\xi}\rfloor \in [0,\xi]$

$$\|e^{-t(\Delta - V_-)}\|_{\infty,\infty} \le \|e^{-\xi(\Delta - V_-)}\|_{\infty,\infty}^{\lfloor \frac{t}{\xi}\rfloor} \|e^{-\delta(\Delta - V_-)}\|_{\infty,\infty} \le C_0^{1+\lfloor \frac{t}{\xi}\rfloor},$$

which gives

$$\|e^{-t(\Delta - V_-)}\|_{\infty,\infty} \le C e^{\alpha_0 t}, \tag{5.6}$$

where the constants C and α_0 depend on C_0.

Step 4 We choose α in **Step 2** sufficiently large so that $\alpha \ge \alpha_0$. Since we have (5.5), the well-known KLMN theorem implies that the operator $\Delta - V_- + \alpha$ is associated to a closed, accretive and symmetric sesquilinear form (see for instance [41] Chapter VI). Therefore, according to [22] Theorem 3.3, the operator $\Delta - V_- + \alpha$ satisfies $L^2 - L^2$ Davies-Gaffney estimates. Furthermore, since the semigroup $(e^{-t(\Delta - V_- + \alpha)})_{t\ge 0}$ is uniformly bounded on $L^\infty(M)$, it is uniformly bounded on $L^1(M)$ by duality. We deduce from [13] Theorem 1.2.1 that the integral kernel $p_t^{V_-}(x,y)$ associated to the semigroup $(e^{-t(\Delta - V_-)})_{t\ge 0}$ satisfies

$$p_t^{V_-}(x,y) \le \frac{Ce^{\alpha t}}{v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right), \forall t > 0, \forall x, y \in M.$$

We conclude using the domination $|\overrightarrow{p_t}(x,y)| \le p_t^{V_-}(x,y)$. □

The following result is (i) of **Theorem 5.1.2**. It is an easy adaptation of [45] Corollaire 2. However we give the proof for the reader's convenience.

Theorem 5.3.2. *Assume (D), (G) and (K). Then the heat kernel on forms $\overrightarrow{p_t}(x,y)$ satisfies for all $t > 0$ and $x, y \in M$*

$$|\overrightarrow{p_t}(x,y)| \le \frac{C\left(1 + \alpha t + \frac{\rho^2(x,y)}{t}\right)^{\frac{D}{2}}}{v(x,\sqrt{t})^{\frac{1}{2}} v(y,\sqrt{t})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right), \tag{5.7}$$

for a certain $\alpha \ge 0$.

Proof. The proof relies on the classical Davies's perturbation method and on the upper bound for $\overrightarrow{p_t}(x,y)$ obtained in **Theorem 5.3.1**. Let $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{C}_0^\infty(M)$ such that $|\nabla\phi| \le 1$ on M. We consider the semigroup $\overrightarrow{T}_{t,\lambda} := e^{-\lambda\phi} e^{-t\overrightarrow{\Delta}} e^{\lambda\phi}$ and its integral kernel

$$\overrightarrow{k}_{t,\lambda}(x,y) = e^{-\lambda(\phi(x)-\phi(y))}\overrightarrow{p_t}(x,y).$$

Step 1 A first consequence of $|\nabla\phi| \leq 1$ and (5.3) is

$$\begin{aligned}|\overrightarrow{k}_{t,\lambda}(x,y)| &\leq \frac{Ce^{\alpha t}e^{|\lambda||\phi(x)-\phi(y)|}}{v(x,\sqrt{t})}\exp\left(-c\frac{\rho^2(x,y)}{t}\right)\\ &\leq \frac{Ce^{\alpha t}}{v(x,\sqrt{t})}\exp\left(-c\frac{\rho^2(x,y)}{t}+|\lambda|\rho(x,y)\right)\\ &\leq \frac{Ce^{\alpha t}}{v(x,\sqrt{t})}\exp\left(\frac{1}{2c}\lambda^2 t\right)\exp\left(-\frac{c}{2}\frac{\rho^2(x,y)}{t}\right).\end{aligned}$$

Step 2 We prove that there exists a constant C independent of λ and ϕ such that for all $t>0$, $x,y\in M$ and $\lambda\in\mathbb{R}$

$$\int_M |\overrightarrow{k}_{\frac{t}{2},\lambda}(x,y)|^2 d\mu(y) \leq \frac{Ce^{\lambda^2 t}}{v\left(x,\sqrt{\min(\frac{t}{2},\frac{1}{\beta})}\right)}, \tag{5.8}$$

where $\beta := (\frac{1}{2c}-1)\lambda^2+\alpha$. One can obviously take the constant c small enough in the estimate of **Step 1** to have $\frac{1}{2c}>1$. Note that if $\beta=0$, then $\lambda=\alpha=0$ and (5.8) follows from the estimate in **Step 1** and **Lemma 5.2.1**. Then in the sequel we suppose $\beta>0$.

We fix $t>0$. According to **Step 1**, if $t\leq\frac{1}{\beta}$

$$\begin{aligned}\int_M |\overrightarrow{k}_{t,\lambda}(x,y)|^2 d\mu(y) &\leq \frac{Ce^{(\frac{1}{c}\lambda^2+2\alpha)t}}{v(x,\sqrt{t})^2}\int_M e^{-c\frac{\rho^2(x,y)}{t}}d\mu(y)\\ &\leq \frac{Ce^{2\beta t}}{v(x,\sqrt{t})}e^{2t\lambda^2} \leq \frac{Ce^2}{v(x,\sqrt{t})}e^{2t\lambda^2}.\end{aligned}$$

Then (5.8) follows for all $t\leq\frac{1}{\beta}$. Now we suppose $t>\frac{1}{\beta}$. The semigroup property implies

$$\int_M |\overrightarrow{k}_{t,\lambda}(x,y)|^2 d\mu(y) \leq \left\|\overrightarrow{T}_{t-\frac{1}{\beta},\lambda}\overrightarrow{k}_{\frac{1}{\beta},\lambda}(x,.)\right\|_2^2 \leq e^{2(t-\frac{1}{\beta})\lambda^2}\left\|\overrightarrow{k}_{\frac{1}{\beta},\lambda}(x,.)\right\|_2^2. \tag{5.9}$$

The last inequality is a consequence of the estimate

$$\|\overrightarrow{T}_{t,\lambda}\|_{2,2} \leq e^{\lambda^2 t}, \forall t\geq 0,$$

which holds since the operator $\overrightarrow{A}_\lambda+\lambda^2$ is positive, where $-\overrightarrow{A}_\lambda$ denotes the generator of the semigroup $(\overrightarrow{T}_{t,\lambda})_{t\geq 0}$. For more details see the proof of [43] Proposition 3.6. It remains to use the inequality

$$\left\|\overrightarrow{k}_{\frac{1}{\beta},\lambda}(x,.)\right\|_2^2 \leq \frac{C}{v(x,\sqrt{\frac{1}{\beta}})}e^{\frac{2\lambda^2}{\beta}},$$

proved above (in the case $t \le \frac{1}{\beta}$) to obtain

$$\left\| \overrightarrow{k}_{t,\lambda}(x,.) \right\|_2^2 \le \frac{C}{v(x, \sqrt{\frac{1}{\beta}})} e^{2t\lambda^2}.$$

This ends the proof of (5.8).

Step 3 We prove that for all $t > 0$ and $x, y \in M$

$$|\overrightarrow{k}_{t,\lambda}(x,y)| \le \frac{Ce^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}}. \tag{5.10}$$

Changing λ into $-\lambda$ in **Step 2** gives the dual inequality

$$\int_M |\overrightarrow{k}_{\frac{t}{2},\lambda}(x,y)|^2 d\mu(x) \le \frac{Ce^{\lambda^2 t}}{v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)}. \tag{5.11}$$

The semigroup property implies

$$|\overrightarrow{k}_{t,\lambda}(x,y)| \le \int_M |\overrightarrow{k}_{\frac{t}{2},\lambda}(x,z)||\overrightarrow{k}_{\frac{t}{2},\lambda}(z,y)| d\mu(z).$$

Thus using the Cauchy-Schwarz inequality, (5.8) and (5.11), we obtain

$$\begin{aligned}
|\overrightarrow{k}_{t,\lambda}(x,y)| &\le \left(\int_M |\overrightarrow{k}_{\frac{t}{2},\lambda}(x,z)|^2 d\mu(z)\right)^{\frac{1}{2}} \left(\int_M |\overrightarrow{k}_{\frac{t}{2},\lambda}(z,y)|^2 d\mu(z)\right)^{\frac{1}{2}} \\
&\le \frac{Ce^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}}.
\end{aligned}$$

Step 4 We prove that for all $t > 0$ and $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \le \frac{C}{v(x, \sqrt{r})^{\frac{1}{2}} v(y, \sqrt{r})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right), \tag{5.12}$$

where

$$r := \min\left(\frac{t}{2}, \left[\left(\frac{1}{2c} - 1\right) \frac{\rho^2(x,y)}{4t^2} + \alpha\right]^{-1}\right).$$

The estimate (5.10) and the definition of $\overrightarrow{k}_{t,\lambda}(x,y)$ give

$$|\overrightarrow{p_t}(x,y)| \leq \frac{Ce^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}} e^{\lambda(\phi(x)-\phi(y))}.$$

Choosing $\lambda = \frac{\phi(y)-\phi(x)}{2t}$, we obtain

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}} \exp\left(-\frac{|\phi(x)-\phi(y)|^2}{4t}\right),$$

with

$$\beta = \left(\frac{1}{2c} - 1\right) \frac{|\phi(x)-\phi(y)|^2}{4t^2} + \alpha.$$

Since $|\nabla\phi| \leq 1$, we have $|\phi(x)-\phi(y)| \leq \rho(x,y)$. We deduce that

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C}{v\left(x, \sqrt{r}\right)^{\frac{1}{2}} v\left(y, \sqrt{r}\right)^{\frac{1}{2}}} \exp\left(-\frac{|\phi(x)-\phi(y)|^2}{4t}\right),$$

with

$$r = \min\left(\frac{t}{2}, \left[\left(\frac{1}{2c} - 1\right)\frac{\rho^2(x,y)}{4t^2} + \alpha\right]^{-1}\right).$$

We obtain (5.12) optimizing in ϕ.

Step 5 We deduce (5.7) using assumption (D). Indeed, noting that

$$v(x, \sqrt{t}) \leq v(x, \sqrt{r}) \left(\frac{t}{r}\right)^{\frac{D}{2}}$$

and

$$\frac{t}{r} \leq 2 + \left(\frac{1}{2c} - 1\right)\frac{\rho^2(x,y)}{4t} + \alpha t,$$

we obtain for all $t > 0$ and $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{2}} v(y,\sqrt{t})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right)\left(1 + \alpha t + \frac{\rho^2(x,y)}{t}\right)^{\frac{D}{2}}.$$

$\square$

The following result has been proved in [23] under different assumptions. Since the ideas of the proof are the same under our assumptions, we just sketch it.

Proposition 5.3.3. *With the same assumptions as in the previous theorem, there exists* $c, C > 0$ *such that for all* $t \geq 1$ *and* $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq C \exp\left(-c\frac{\rho^2(x,y)}{t}\right).$$

Proof. Following the proof of [23] Theorem 4.1 Step 1, we find for $A > 0$ sufficiently large and $t \geq 1$

$$\int_M |\overrightarrow{p_t}(x,y)|^2 e^{\frac{\rho^2(x,y)}{At}} d\mu(x) \leq \int_M |\overrightarrow{p_1}(x,y)|^2 e^{\frac{\rho^2(x,y)}{A}} d\mu(x).$$

The estimate (5.7) gives

$$|\overrightarrow{p_1}(x,y)| \leq \frac{C}{v(x,1)} e^{-c\rho^2(x,y)}.$$

We deduce that for $A > 0$ sufficiently large and all $t \geq 1$

$$\int_M |\overrightarrow{p_t}(x,y)|^2 e^{\frac{\rho^2(x,y)}{At}} d\mu(x) \leq C.$$

Then the rest of the proof is the same as in [23] Theorem 4.1 Step 1. □

We deduce from **Theorem 5.3.2** and **Proposition 5.3.3** the following result which is (ii) of **Theorem 5.1.2**.

Corollary 5.3.4. *Assume (D), (G) and (K). Then there exists* $c, C > 0$ *such that for all* $t \geq 1$ *and* $x, y \in M$

$$|\overrightarrow{p_t}(x,y)| \leq C \min\left(1, \frac{t^{\frac{D}{2}}}{v(x,\sqrt{t})}\right) \exp\left(-c\frac{\rho^2(x,y)}{t}\right). \tag{5.13}$$

Now we prove (iii) of **Theorem 5.1.2**. A related result can be found in [21] where the authors supposed that $\overrightarrow{p_t}(x,y)$ satisfies a Gaussian upper bound instead of (5.7). However the main part of the proof is known, we write it for the sake of completeness.

Corollary 5.3.5. *Assume (D), (G) and (K). Then for all* $2 \leq p \leq \infty$ *and* $t \geq 1$,

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C t^{\frac{D}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}.$$

Proof. **Step 1** We prove that there exists constants $c, C > 0$ such that

$$\forall x, y \in M \forall t > 0, \quad \left|\frac{\partial}{\partial t} p_t(x,y)\right| \leqslant \frac{C}{t\, v(x,\sqrt{t})} e^{-c\frac{\rho^2(x,y)}{t}}.$$

The proof relies on ideas of Davies [24]. We consider $\lambda \in \mathbb{R}$ and $\varphi \in W^{1,\infty}(M)$ such that $|\nabla\varphi| \leqslant 1$. We denote $S_\lambda(t) = e^{\lambda\varphi} e^{-t\Delta} e^{-\lambda\varphi}$ for all $t > 0$ and its integral kernel

$$K_{\lambda,t}(x,y) = e^{\lambda(\varphi(x)-\varphi(y))} p_t(x,y),$$

Since we assume (G), we have

$$\begin{aligned}
|K_{\lambda,t}(x,y)| &\leqslant e^{|\lambda||\varphi(x)-\varphi(y)|} \frac{C}{v(x,\sqrt{t})} e^{-c\frac{\rho^2(x,y)}{t}} \\
&\leqslant \frac{C}{v(x,\sqrt{t})} e^{\frac{2\lambda^2 t}{c}} e^{c\frac{|\varphi(x)-\varphi(y)|^2}{2t}} e^{-c\frac{\rho^2(x,y)}{t}} \\
&\leqslant \frac{C}{v(x,\sqrt{t})} e^{\frac{2\lambda^2 t}{c}} e^{-c\frac{\rho^2(x,y)}{2t}}.
\end{aligned}$$

We denote $-\Delta_\lambda = -e^{\lambda\varphi}\Delta e^{-\lambda\varphi}$ the generator of the semigroup $(S_\lambda(t))_{t>0}$. We prove that the operator $\sqrt{v(.,\sqrt{t})}(\Delta_\lambda)S_\lambda(t)\sqrt{v(.,\sqrt{t})}$ is bounded from $L^1(M)$ to $L^\infty(M)$. We write

$$\sqrt{v(.,\sqrt{t})}(\Delta_\lambda)S_\lambda(t)\sqrt{v(.,\sqrt{t})} = \sqrt{v(.,\sqrt{t})}S_\lambda(\frac{t}{3})\ (\Delta_\lambda)S_\lambda(\frac{t}{3})\ S_\lambda(\frac{t}{3})\sqrt{v(.,\sqrt{t})}.$$

The operator $\sqrt{v(.,\sqrt{t})}S_\lambda(\frac{t}{3})$ has for integral kernel $\sqrt{v(.,\sqrt{t})}K_{\lambda,\frac{t}{3}}(x,y)$. Using **Lemma 5.2.1** and the above estimate of $|K_{\lambda,t}(x,y)|$, it is easy to prove that there exist constants $c, C > 0$ independent of λ and φ such that

$$\sup_{x\in M}\int_M |\sqrt{v(x,\sqrt{t})}K_{\lambda,\frac{t}{3}}(x,y)|^2 d\mu(y) \leqslant Ce^{c\lambda^2 t}.$$

Therefore the operator $\sqrt{v(.,\sqrt{t})}S_\lambda(\frac{t}{3})$ is bounded from $L^2(M)$ to $L^\infty(M)$ with

$$\|\sqrt{v(.,\sqrt{t})}S_\lambda(\frac{t}{3})\|_{2,\infty} \leqslant \sqrt{Ce^{c\lambda^2 t}}.$$

The same arguments together with assumption (D) lead to

$$\sup_{y\in M}\int_M |K_{\lambda,\frac{t}{3}}(x,y)\sqrt{v(y,\sqrt{t})}|^2 d\mu(x) \leqslant Ce^{c\lambda^2 t},$$

so that the operator $S_\lambda(\frac{t}{3})\sqrt{v(.,\sqrt{t})}$ is bounded from $L^1(M)$ to $L^2(M)$ with

$$\|S_\lambda(\frac{t}{3})\sqrt{v(.,\sqrt{t})}\|_{1,2} \leqslant \sqrt{Ce^{c\lambda^2 t}}.$$

Furthermore using the theory of sesquilinear forms, one can prove that the operator $\Delta_\lambda + 2\lambda^2$ is sectorial and then that the semigroup $(S_\lambda(t)e^{-2\lambda^2 t})_{t>0}$ is analytic on $L^2(M)$. We deduce from the Cauchy formula that the operator $\Delta_\lambda S_\lambda(\frac{t}{3})$ is bounded on $L^2(M)$ with

$$\|(\Delta_\lambda)S_\lambda(\frac{t}{3})\|_{2,2} \leqslant \frac{C}{t}e^{2\lambda^2 t},$$

with a constant $C > 0$ independent of λ and φ. We conclude that $\sqrt{v(.,\sqrt{t})}(\Delta_\lambda)S_\lambda(t)\sqrt{v(.,\sqrt{t})}$ is bounded from $L^1(M)$ to $L^\infty(M)$ with

$$\|\sqrt{v(.,\sqrt{t})}(\Delta_\lambda)S_\lambda(t)\sqrt{v(.,\sqrt{t})}\|_{1,\infty} \leqslant \frac{C'}{t}e^{c'\lambda^2 t}.$$

Since the integral kernel of the operator $\sqrt{v(.,\sqrt{t})}(\Delta_\lambda)S_\lambda(t)\sqrt{v(.,\sqrt{t})}$ is

$$-\sqrt{v(x,\sqrt{t})}\sqrt{v(y,\sqrt{t})}e^{\lambda(\varphi(x)-\varphi(y))}\frac{\partial}{\partial t}p_t(x,y),$$

the Dunford-Pettis theorem ensures that

$$\left|\frac{\partial}{\partial t}p_t(x,y)e^{\lambda(\varphi(x)-\varphi(y))}\right| \leqslant \frac{C'}{t\sqrt{v(x,\sqrt{t})}\sqrt{v(y,\sqrt{t})}}e^{c'\lambda^2 t},$$

that is

$$\left|\frac{\partial}{\partial t}p_t(x,y)\right| \leqslant \frac{C'}{t\sqrt{v(x,\sqrt{t})}\sqrt{v(y,\sqrt{t})}}e^{c'\lambda^2 t}e^{\lambda(\varphi(y)-\varphi(x))}$$

We choose $\lambda = \frac{\varphi(x)-\varphi(y)}{\gamma t}$ with $\gamma > c'$ so that

$$\left|\frac{\partial}{\partial t}p_t(x,y)\right| \leqslant \frac{C'}{t\sqrt{v(x,\sqrt{t})}\sqrt{v(y,\sqrt{t})}}e^{-c''\frac{(\varphi(y)-\varphi(x))^2}{t}}$$

Since this estimate holds for all function $\varphi \in W^{1,\infty}(M)$ satisfying $|\nabla\varphi| \leqslant 1$, we obtain

$$\left|\frac{\partial}{\partial t}p_t(x,y)\right| \leqslant \frac{C'}{t\sqrt{v(x,\sqrt{t})}\sqrt{v(y,\sqrt{t})}}e^{-c''\frac{\rho^2(x,y)}{t}}.$$

It remains to use assumption (D) to obtain the estimate of **Step 1**.

Step 2 We prove that for all $\alpha > 0$ small enough

$$\int_M |\nabla_x p_t(x,y)|^2 e^{\alpha\frac{\rho^2(x,y)}{t}}d\mu(x) \leqslant \frac{C_\alpha}{t\,v(y,\sqrt{t})},\ \forall y \in M, \forall t > 0.$$

Integrating by parts gives

$$\begin{aligned}
I(t,y) &= \int_M |\nabla_x p_t(x,y)|^2 e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x) \\
&= \int_M p_t(x,y)\Delta_x p_t(x,y) e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x) \\
&\quad - \int_M p_t(x,y)\nabla_x p_t(x,y)\nabla_x (e^{\alpha\frac{\rho^2(x,y)}{t}}) d\mu(x) \\
&= I_1(t,y) + I_2(t,y).
\end{aligned}$$

Notice that

$$I_1(t,y) = -\int_M p_t(x,y)\frac{\partial}{\partial t} p_t(x,y) e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x).$$

Therefore using **Step 1** and assumptions (D) and (G), we find

$$\begin{aligned}
I_1(t,y) &\leqslant \frac{C}{t\, v(y,\sqrt{t})^2}\int_M e^{-c\frac{\rho^2(x,y)}{t}} e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x) \\
&\leqslant \frac{C_\alpha}{t\, v(y,\sqrt{t})},
\end{aligned}$$

where we take $\alpha < 2c$ and use **Lemma 5.2.1** to obtain the last inequality. Now we estimate $I_2(t,y)$. We have

$$I_2(t,y) = -\int_M p_t(x,y)\nabla_x p_t(x,y)\frac{2\alpha\rho(x,y)}{t}\nabla_x\rho(x,y) e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x).$$

Since $|\nabla_x \rho(x,y)| \leqslant 1$, we obtain

$$\begin{aligned}
|I_2(t,y)| &\leqslant \int_M p_t(x,y)|\nabla_x p_t(x,y)|\frac{2\alpha\rho(x,y)}{t} e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x) \\
&\leqslant \frac{C}{\sqrt{t}}\int_M p_t(x,y)|\nabla_x p_t(x,y)| e^{(\alpha+\alpha')\frac{\rho^2(x,y)}{t}} d\mu(x) \quad (\text{since } \frac{\rho(x,y)}{\sqrt{t}} \leqslant C e^{\alpha'\frac{\rho^2(x,y)}{t}}) \\
&\leqslant \frac{C}{\sqrt{t}}\left(\int_M |p_t(x,y)|^2 e^{(\alpha+2\alpha')\frac{\rho^2(x,y)}{t}} d\mu(x)\right)^{\frac{1}{2}}\left(\int_M |\nabla_x p_t(x,y)|^2 e^{\alpha\frac{\rho^2(x,y)}{t}} d\mu(x)\right)^{\frac{1}{2}},
\end{aligned}$$

from the Cauchy-Schwarz inequality. Taking α and α' small enough and using (G), (D) and **Lemma 5.2.1** lead to

$$\begin{aligned}
\int_M |p_t(x,y)|^2 e^{(\alpha+2\alpha')\frac{\rho^2(x,y)}{t}} d\mu(x) &\leqslant \frac{C}{v(y,\sqrt{t})^2}\int_M e^{-2c\frac{\rho^2(x,y)}{t}} e^{(\alpha+2\alpha')\frac{\rho^2(x,y)}{t}} d\mu(x) \\
&\leqslant \frac{C'}{v(y,\sqrt{t})},
\end{aligned}$$

Therefore

$$|I_2(t,y)| \leqslant \frac{C}{\sqrt{t\,v(y,\sqrt{t})}}\sqrt{I(t,y)}.$$

We deduce that

$$\begin{aligned}
I(t,y) &\leqslant \frac{C}{t\,v(y,\sqrt{t})} + \frac{C'}{\sqrt{t\,v(y,\sqrt{t})}}\sqrt{I(t,y)} \\
&\leqslant \frac{C}{t\,v(y,\sqrt{t})} + \frac{C'^2}{2t\,v(y,\sqrt{t})} + \frac{I(t,y)}{2},
\end{aligned}$$

which gives the result of **Step 2**.

Step 3 We prove that there exists $\gamma > 0$ and $C_\gamma > 0$ such that

$$\int_M |\nabla_x p_t(x,y)| e^{\gamma \frac{\rho^2(x,y)}{t}} d\mu(x) \leqslant \frac{C_\gamma}{\sqrt{t}},\ \forall y \in M, \forall t > 0.$$

The Cauchy-Schwarz inequality ensures that for all $y \in M$ and $s,t > 0$

$$\begin{aligned}
&\int_M |\nabla_x p_t(x,y)| e^{\gamma \frac{\rho^2(x,y)}{t}} d\mu(x) \\
&\leqslant \left(\int_M |\nabla_x p_t(x,y)|^2 e^{4\gamma \frac{\rho^2(x,y)}{t}} d\mu(x)\right)^{\frac{1}{2}} \left(\int_M e^{-2\gamma \frac{\rho^2(x,y)}{t}} d\mu(x)\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{C_\gamma}{t\,v(y,\sqrt{t})}\right)^{\frac{1}{2}} (C'_\gamma v(y,\sqrt{t}))^{\frac{1}{2}} \\
&= \frac{C''_\gamma}{\sqrt{t}},
\end{aligned}$$

where we used the result of **Step 2** (with $\gamma > 0$ small enough) and **Lemma 5.2.1** to obtain the last inequality.

Step 4 We prove that there exists $c, C > 0$ such that

$$|\nabla_x p_t(x,y)| \leqslant \frac{C t^{\frac{D}{2}}}{\sqrt{t}\,v(x,\sqrt{t})} e^{-c\frac{\rho^2(x,y)}{t}},\ \forall x,y \in M, \forall t > 1.$$

First notice that

$$\nabla_x p_t(x,y) = \int_M \nabla_x p_{\frac{t}{2}}(x,z) p_{\frac{t}{2}}(z,y) d\mu(z) = \nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x),$$

with $f_{y,t}(z) = p_{\frac{t}{2}}(z,y)$. According to the commutation formula $\overrightarrow{\Delta} d = d\Delta$, we have

$$|\nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x)| = |e^{\frac{t}{2}\overrightarrow{\Delta}} df_{y,t}(x)|.$$

Therefore using the estimate (5.7) and (D), we obtain for all $t > 1$

$$\begin{aligned}|\nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x)| &\leqslant \frac{Ct^{\frac{D}{2}}}{v(x,\sqrt{t})} \int_M e^{-2c\frac{\rho^2(x,z)}{t}} |df_{y,t}(z)| d\mu(z) \\ &= \frac{Ct^{\frac{D}{2}}}{v(x,\sqrt{t})} \int_M e^{-2c\frac{\rho^2(x,z)}{t}} |\nabla_z f_{y,t}(z)| d\mu(z).\end{aligned}$$

Hence

$$|\nabla_x p_t(x,y)| \leqslant \frac{Ct^{\frac{D}{2}}}{v(x,\sqrt{t})} \int_M e^{-2c\frac{\rho^2(x,z)}{t}} |\nabla_z f_{y,t}(z)| d\mu(z).$$

It suffices then to prove that for some $\gamma < 2c$, there exist constants $c', C' > 0$ such that

$$\int_M e^{-\gamma\frac{\rho^2(x,z)}{t}} |\nabla_z p_{\frac{t}{2}}(z,y)| d\mu(z) \leqslant \frac{C'}{\sqrt{t}} e^{-c'\frac{\rho^2(x,y)}{t}}.$$

Using

$$e^{-\gamma\frac{\rho^2(x,z)}{t}} \leqslant e^{-\gamma\frac{\rho^2(x,y)}{2t}} e^{\gamma\frac{\rho^2(z,y)}{t}},$$

we have

$$\int_M e^{-\gamma\frac{\rho^2(x,z)}{t}} |\nabla_z p_{\frac{t}{2}}(z,y)| d\mu(z) \leqslant e^{-\gamma\frac{\rho^2(x,y)}{2t}} \int_M e^{\gamma\frac{\rho^2(z,y)}{t}} |\nabla_z p_{\frac{t}{2}}(z,y)| d\mu(z).$$

According to **Step 3**, we know that there exists $\gamma > 0$ small enough such that

$$\int_M e^{\gamma\frac{\rho^2(x,z)}{t}} |\nabla_z p_{\frac{t}{2}}(z,y)| d\mu(z) \leqslant \frac{C'}{\sqrt{t}},$$

which proves the claim of **Step 4**.
Step 5 It is well-known that for all $t > 0$

$$\|\nabla e^{-t\Delta}\|_{2,2} \leq \frac{C}{\sqrt{t}}. \tag{5.14}$$

It suffices then to prove that for all $t \geq 1$

$$\|\nabla e^{-t\Delta}\|_{\infty,\infty} \leq \frac{Ct^{\frac{D}{2}}}{\sqrt{t}}, \tag{5.15}$$

and use classical interpolation between (5.14) and (5.15). Note that (5.15) is an immediate consequence of **Step 4** integrating over M and using **Lemma 5.2.1**. $\square$

Finally the proof of **Theorem 5.1.2** (iv) is a straightforward adaptation of [45] Théorème 7 using (5.3). The reader can also find details in the proof of **Proposition 5.4.3** below (in the case $p = 2$).

It remains to prove **Corollary 5.1.3**.

*Proof of **Corollary 5.1.3**.* Fix $a > 0$. From (5.7), we deduce that there exists $c_a, C_a > 0$ such that for all $t > 0$

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C_a e^{at}}{v(x,\sqrt{t})} \exp\left(-c_a \frac{\rho^2(x,y)}{t}\right).$$

It follows from [48] that the Riesz transform $d^*(\overrightarrow{\Delta}+a)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in (1,2]$. A duality argument based on the commutation formula $\overrightarrow{\Delta} d = d\Delta$ implies that the local Riesz transform $d(\Delta + a)^{-\frac{1}{2}}$ is bounded on L^p for all $p \in [2,\infty)$. The case $p \in (1,2]$ is proved in [20] under the assumptions (D) and (G). $\square$

5.4 Proof of Theorem 5.1.4

In this section, we still assume (D), (G) and (K). In addition, we suppose that R_- is ϵ-subcritical, that is there exists $\epsilon \in [0,1)$ such that

$$0 \leq (R_-\omega,\omega) \leq \epsilon\,(H\omega,\omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), \tag{S-C}$$

where $H = \nabla^*\nabla + R_+$. We show how to improve the results of **Theorem 5.1.2**. We denote $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$.

We start by proving (i) of **Theorem 5.1.4**. The proof is based on ideas in [45] and on results in [2], [43], [13] and [48].

Theorem 5.4.1. *Assume that (D), (G), (K) and (S-C) are satisfied. Then for all $t > 0$ and $x, y \in M$*

$$|\overrightarrow{p_t}(x,y)| \leq \frac{C(1+t)^{\frac{D}{\tilde{p}_0}}}{v(x,\sqrt{t})} \exp\left(-c\frac{\rho^2(x,y)}{t}\right), \tag{5.16}$$

where $\tilde{p}_0 = p_0 - \eta$ for any fixed small $\eta > 0$.

Proof. We fix a small $\eta > 0$.

Step 1 We show the $L^2 - L^p$ estimates

$$\sup_{t>0} \|e^{-t\overrightarrow{\Delta}} v(.,\sqrt{t})^{\frac{1}{2}-\frac{1}{p}}\|_{2,p} \leq C \tag{5.17}$$

for all $p \in [2, p_0)$.
Let $k \in \mathbb{N}$ and let $A(x, \sqrt{t}, k)$ denote the annulus $B(x, (k+1)\sqrt{t}) \setminus B(x, k\sqrt{t})$. Following the proof of [43] Theorem 4.1 leads to the following $L^q - L^2$ off-diagonal estimates

$$\|\chi_{B(x,\sqrt{t})} e^{-t\overrightarrow{\Delta}} \chi_{A(x,\sqrt{t},k)}\|_{q,2} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{q}-\frac{1}{2}}} e^{-ck^2}$$

for all $q \in (p_0', 2]$. Then following the proof of [2] Proposition 2.9, we obtain for all $q \in (p_0', 2]$

$$\sup_{t>0} \|v(.,\sqrt{t})^{\frac{1}{q}-\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{q,2} \leq C.$$

We deduce (5.17) by duality.

Step 2 We prove that for all $t > 0$

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq C(1+t)^{\frac{D}{2\tilde{p}_0}}. \tag{5.18}$$

Let $0 < t \leq 1$. Using (5.3) and **Lemma 5.2.1**, we easily obtain

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq C \leq C(1+t)^{\frac{D}{2\tilde{p}_0}}.$$

We now consider $t > 1$. Since $\overrightarrow{\Delta}$ satisfies the $L^2 - L^2$ Davies-Gaffney estimates ([48] Theorem 6), a consequence of [13] Proposition 4.1.6 is

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq C\|v(.,\sqrt{t})^{\frac{1}{\tilde{p}_0}} e^{-t\overrightarrow{\Delta}} v(.,\sqrt{t})^{\frac{1}{2}-\frac{1}{\tilde{p}_0}}\|_{2,\infty},$$

with C independent of t. The semigroup property then gives

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq C\|v(.,\sqrt{t})^{\frac{1}{\tilde{p}_0}} e^{-\frac{t}{2}\overrightarrow{\Delta}}\|_{\tilde{p}_0,\infty} \|e^{-\frac{t}{2}\overrightarrow{\Delta}} v(.,\sqrt{t})^{\frac{1}{2}-\frac{1}{\tilde{p}_0}}\|_{2,\tilde{p}_0}.$$

We use (5.17) and (D) to obtain

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq C\|v(.,\sqrt{t/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{t}{2}\overrightarrow{\Delta}}\|_{\tilde{p}_0,\infty}.$$

Using again (D) and the semigroup property leads to

$$\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} \leq Ct^{\frac{D}{2\tilde{p}_0}} \|v(.,\sqrt{1/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{1}{2}\overrightarrow{\Delta}}\|_{\tilde{p}_0,\infty} \|e^{-(\frac{t}{2}-\frac{1}{2})\overrightarrow{\Delta}}\|_{\tilde{p}_0,\tilde{p}_0}.$$

According to [43] Theorem 3.1 we know that the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is uniformly bounded on $L^{\tilde{p}_0}$. Therefore

$$\begin{aligned}\|v(.,\sqrt{t})^{\frac{1}{2}} e^{-t\overrightarrow{\Delta}}\|_{2,\infty} &\leq Ct^{\frac{D}{2\tilde{p}_0}} \|v(.,\sqrt{1/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{1}{2}\overrightarrow{\Delta}}\|_{\tilde{p}_0,\infty} \\ &\leq C(1+t)^{\frac{D}{2\tilde{p}_0}},\end{aligned}$$

where the last inequality is an easy consequence of (5.3) and **Lemma 5.2.1**. This concludes the proof of (5.18).

Step 3 We end the proof using [48] Theorem 1 and applying [48] Theorem 4 with the function

$$V_z(t) := \frac{C(1+t)^{\frac{D}{\tilde{p}_0}}}{v(z,t)^{\frac{1}{2}}}.$$

□

Now (ii) of **Theorem 5.1.4** is an immediate consequence of (5.16) for $t \geq 1$ and **Proposition 5.3.3**.

The next statement aims at proving (iii) of **Theorem 5.1.4**.

Corollary 5.4.2. *Assume that (D), (G), (K) and (S-C) are satisfied. Then for all* $t \geq 1$

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq \frac{C_p}{\sqrt{t}} \text{ if } p \in (1, p_0)$$

and

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C_p t^{\left(\frac{1}{\tilde{p}_0} - \frac{1}{p}\right)D - \frac{1}{2}} \text{ if } p \geq p_0.$$

Proof. The case $p \in (1,2]$ is well-known and the proof only uses assumptions (D) and (G). Let $p > 2$. Taking back the proof of **Corollary 5.3.5** and using (5.16) instead of (5.7) in **Step 4**, one can obtain for all $t \geq 1$

$$|\nabla_x p_t(x,y)| \leq \frac{Ct^{\frac{D}{\tilde{p}_0}}}{\sqrt{t}} \exp\left(-c\frac{\rho^2(x,y)}{t}\right)$$

and

$$\|\nabla e^{-t\Delta}\|_{\infty,\infty} \leq \frac{Ct^{\frac{D}{\tilde{p}_0}}}{\sqrt{t}}. \tag{5.19}$$

Furthermore [43] Corollary 1.2 gives that the Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded on $L^{\tilde{p}_0}$. Since the semigroup $(e^{-t\Delta})_{t\geq 0}$ is bounded analytic on $L^{\tilde{p}_0}$, we can deduce from the Cauchy formula that

$$\|\nabla e^{-t\Delta}\|_{\tilde{p}_0,\tilde{p}_0} \leq \frac{C}{\sqrt{t}}. \tag{5.20}$$

Therefore the first and second part of **Corollary 5.4.2** can be obtained interpolating between (5.14) and (5.19) and between (5.19) and (5.20) respectively. □

It remains to prove (iv) of **Theorem 5.1.4**. This is the purpose of the next two propositions. We start with a slightly more general version of [45] Théorème 7.

Proposition 5.4.3. *Assume that (D), (G) and (K) are satisfied. Assume in addition that the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is uniformly bounded on L^p for some $p \in (1,2]$. Then for all $t > e$*

$$\|e^{-t\overrightarrow{\Delta}}\|_{\infty,\infty} \leq C(t\log(t))^{\frac{D}{2}\left(1-\frac{1}{p}\right)}.$$

Proof. Let $t > e$. Since the semigroup $(e^{-t\overrightarrow{\Delta}})_{t\geq 0}$ is uniformly bounded on L^p, we have

$$\int_M |\overrightarrow{p_t}(x,y)|^p d\mu(y) \leq \|e^{-(t-1)\overrightarrow{\Delta}}\|_{p,p}^p \, \||\overrightarrow{p_1}(x,.)|\|_p^p \leq C\int_M |\overrightarrow{p_1}(x,y)|^p d\mu(y).$$

Then using (5.7) and **Lemma 5.2.1** we find

$$\int_M |\overrightarrow{p_1}(x,y)|^p d\mu(y) \leq \frac{C}{v(x,1)^p}\int_M \exp\left(-cp\rho^2(x,y)\right) \leq \frac{C}{v(x,1)^{p-1}}.$$

Hence

$$\int_M |\overrightarrow{p_t}(x,y)|^p d\mu(y) \leq \frac{C}{v(x,1)^{p-1}}. \tag{5.21}$$

Let $\epsilon \in (0,1)$ and $p_\epsilon \in (0,1)$ such that $\frac{1-\epsilon}{p} + \frac{\epsilon}{p_\epsilon} = 1$, that is $p_\epsilon = \frac{\epsilon p}{p-1+\epsilon}$. Using (5.7) and **Lemma 5.2.1**, we estimate

$$\int_M |\overrightarrow{p_t}(x,y)|^{p_\epsilon} d\mu(y) \leq \frac{Ct^{\frac{Dp_\epsilon}{2}}}{v(x,\sqrt{t})^{p_\epsilon}}\int_M \exp\left(-cp_\epsilon\frac{\rho^2(x,y)}{t}\right)d\mu(y) \leq \frac{Ct^{\frac{Dp_\epsilon}{2}}}{v(x,\sqrt{t})^{p_\epsilon}}v(x,\sqrt{\frac{t}{p_\epsilon}}).$$

Using (D) we obtain

$$\int_M |\overrightarrow{p_t}(x,y)|^{p_\epsilon} d\mu(y) \leq Ct^{\frac{Dp_\epsilon}{2}} v(x,\sqrt{t})^{1-p_\epsilon} p_\epsilon^{-\frac{D}{2}}. \tag{5.22}$$

From (5.21), (5.22) and the Hölder inequality, we deduce that

$$\begin{aligned}
\int_M |\overrightarrow{p_t}(x,y)| d\mu(y) &\leq \left(\int_M |\overrightarrow{p_t}(x,y)|^p d\mu(y)\right)^{\frac{1-\epsilon}{p}} \left(\int_M |\overrightarrow{p_t}(x,y)|^{p_\epsilon} d\mu(y)\right)^{\frac{\epsilon}{p_\epsilon}} \\
&\leq C\left(\frac{v(x,\sqrt{t})}{v(x,1)}\right)^{\frac{(p-1)(1-\epsilon)}{p}} t^{\frac{\epsilon D}{2}} p_\epsilon^{-\frac{D(p-1+\epsilon)}{2p}} \\
&\leq Ct^{\frac{D(p-1)}{2p}} t^{\frac{\epsilon D}{2p}} p_\epsilon^{-\frac{D(p-1+\epsilon)}{2p}},
\end{aligned}$$

where (D) has been used to obtain the last inequality. Note that from the definition of p_ϵ, we have $p_\epsilon^{-(p-1+\epsilon)} \le C\epsilon^{1-p}$. Hence

$$\int_M |\overrightarrow{p_t}(x,y)|d\mu(y) \le Ct^{\frac{D(p-1)}{2p}} \left[t^\epsilon \epsilon^{1-p}\right]^{\frac{D}{2p}}. \tag{5.23}$$

Noticing that the RHS has its minimum for $\epsilon = \frac{p-1}{\log(t)} \in (0,1)$ (since $t > e$), we conclude that

$$\int_M |\overrightarrow{p_t}(x,y)|d\mu(y) \le C(t\log(t))^{\frac{D(p-1)}{2p}},$$

which is the desired result. $\square$

Next we recall a consequence of [43] Theorem 3.1.

Proposition 5.4.4. *Assume that (D), (G) and (S-C) are satisfied. Then the semigroup* $(e^{-t\overrightarrow{\Delta}})_{t\ge 0}$ *is uniformly bounded on* L^p *for all* $p \in (p_0', p_0)$.

Therefore (iv) of **Theorem 5.1.4** is obtained by combining **Proposition 5.4.4** and **Proposition 5.4.3**, and using classical interpolation and duality.

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